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The multiple steiner TSP with cyclic order on terminals: valid inequalities and polyhedra

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Abstract

This paper deals with a variant of the Traveling Salesman Problem (TSP), called the Multiple Steiner TSP with Order Constraints (MSTSPOC). Consider an undirected graph with nonnegative weights on the edges, and a set of salesmen such that with each salesman is associated a set of ordered terminals. The MSTSPOC consists in finding a minimum-weight subgraph containing for each salesman a tour going in order through its terminals. We study the polytope associated with the Integer Linear Programming (ILP) formulation proposed in Borne et al. (2013). We characterize when the basic inequalities define facets. We also describe new valid inequalities along with necessary conditions and sufficient conditions for these inequalities to be facet-defining. Further families of valid inequalities, coming from closely related problems, are also discussed. The theoretical results presented in this paper are computationally tested in a companion paper (Taktak 2024).

Keywords Multiple Steiner TSP · Order constraints · Polytope · Facet · Valid inequality

Mathematics Subject Classification 90C10 · 52B05

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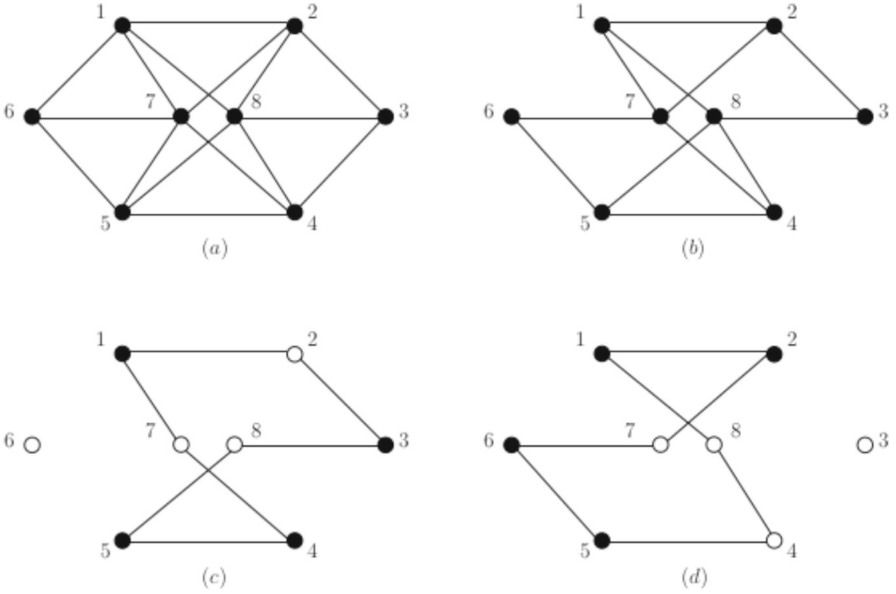

Fig. 1 Illustrative example

1 Introduction and related works

We consider a variant of the Traveling Salesman Problem (TSP), called the Multiple Steiner Traveling Salesman Problem with Order Constraints (MSTSPOC). Let $G = (V, E)$ be an undirected graph with nonnegative weights on the edges and K a set of salesmen. For each salesman $k \in K$, there is a set $T_k \subseteq V$ of *terminals*. The MSTSPOC consists in finding a set of edges $F \subseteq E$, with minimum total weight, such that for each salesman $k \in K$ there is a *Steiner tour* that visits all terminals in T_k in a predefined cyclic order. Steiner nodes not belonging to T_k are optional and are denoted by S_k. These nodes may be traversed between visits to terminals. A *Steiner tour* is a cycle going once through each of its terminals. In the sequel, we may simply say tour to refer to a Steiner tour. Moreover, due to some survivability restrictions, tours must be *elementary*, that is nodes and edges are not allowed to be visited more than once (see Borne et al. 2011, 2013).

In Fig. 1, we give an example in order to illustrate the MSTSPOC. Consider the graph in Fig. 1(a). The graph is composed of 8 nodes numbered from 1 to 8. Some of these nodes are connected by edges as shown in the figure. Assume that $|K| = 2$, and that $T_1 = \{1, 3, 5, 4\}$ and $T_2 = \{1, 2, 6, 5\}$ both must be visited in the given order, and both come back to node 1. Note here that, according to the notations above mentioned, Steiner nodes for both salesmen are $S_1 = \{2, 6, 7, 8\}$ and $S_2 = \{3, 4, 7, 8\}$, respectively. A feasible solution for the MSTSPOC is illustrated in Fig. 1(b). The tours of both salesmen are illustrated in Fig. 1(c) and Fig. 1(d), respectively. The first salesman will have the following tour (1, 2, 3, 8, 5, 4, 7, 1), and the second one, (1, 2, 7, 6, 5, 4, 8, 1).

1.1 Previous works and our contributions

The MSTSPOC was first introduced in Borne et al. (2011) motivated by reliability issues in multilayer telecommunication networks. Several ILP formulations have been then proposed for the MSTSPOC, and a variety of algorithms has also been devised to solve the problem. In Borne et al. (2011), the authors prove that the problem is NP-hard even for a single salesman. The problem is reduced to the k-Vertex Disjoint Paths Problem, known to be NP-hard. In Borne et al. (2011), the authors give a path-based formulation for the MSTSPOC, and present preliminary experimental results. In a further work (Gabrel et al. 2020), the authors investigate more the path-based formulation. The pricing problem is discussed, specific branching strategies are described, and a Branch-and-Price algorithm is devised. A substantial experimental study is also presented. In Mahjoub et al. (2019), the authors propose a compact ILP formulation for the MSTSPOC. In this formulation, the tour of each salesman is viewed as a union of layers, each characterized by two terminals. The authors present experimental results that show the efficiency of the corresponding model. A third ILP formulation, called cut-formulation, is also proposed for the MSTSPOC. It is first introduced in Borne et al. (2013), where the authors present some preliminary results of a Branch-and-Cut algorithm. The cut formulation is then strengthened using several families of valid inequalities in Taktak and Uchoa (2020), where the authors present briefly some theoretical results related to the polytope associated with this formulation. In Taktak (2024), the author devise a Branch-and-Cut algorithm for the cut formulation to solve the MSTSPOC. In particular, separation routines are proposed for the different classes of valid inequalities used in the algorithm. Moreover, experimental results are presented, and they show the efficiency of the valid inequalities in solving random and realistic instances. Note that the work in Taktak (2024) is complementary to the current work since both concern the cut formulation. In fact, paper (Taktak 2024) is experimental and aims at giving the experimental proofs of the efficiency of the valid inequalities whose facial aspect is studied in the current paper.

In this paper, we study the polytope of the cut formulation, that is the convex hull of the solutions of the cut formulation. We first recall the cut formulation introduced in Borne et al. (2013). Then we characterize the dimension of the polytope and investigate the facial structure of the basic inequalities. We also describe new classes of valid inequalities and give necessary conditions and sufficient conditions for these inequalities to be facet-defining. The efficiency of these valid inequalities in solving the MSTSPOC is shown in Taktak (2024). Finally, we describe further classes of valid inequalities that extend valid inequalities coming from relaxed problems like the k-Node Connected Subgraph Problem, the k-Edge Connected Subgraph Problem, and the TSP.

1.2 Other related works

Several variants of the TSP are closely related to the MSTSPOC. In what follows, we review some of them.

The *Steiner Traveling Salesman Problem (STSP)* is a variant of the TSP in which only a given subset of nodes, called terminals, must be visited in a minimum-weight cycle. The cycle may contain non terminal nodes, called *Steiner* nodes. Letchford et al. (2013) propose several compact formulations for the STSP obtained from ones known for the TSP. The authors compare these formulations both theoretically and computationally. In Interian and Ribeiro (2017), Interian and Ribeiro adapt some classical constructive heuristics as well as neighborhood structures used for the TSP to solve the STSP. In particular, they devise a GRASP heuristic-based algorithm that proved to be efficient for solving large-sized instances. In Rodríguez-Pereira et al. (2019), Rodríguez-Pereira et al. consider the STSP as well as some extensions with depots. They propose compact ILP formulations for the studied variants of the STSP, and devise Branch-and-Cut algorithms. Álvarez-Miranda and Sinnl (2019) propose a transformation of the STSP to the classical TSP. In consequence, they use a state-of-the-art TSP-solver to solve a benchmark of instances. Compared to the approaches developed in Letchford et al. (2013) and Interian and Ribeiro (2017), the authors were able to solve all instances from the literature to optimality within 20 seconds. In particular, they provide optimal solutions for 14 instances not yet solved.

A further interesting variant is the so-called *multiple TSP* (mTSP) which consists in finding a set of tours for a predefined number of salesmen, each starting from and coming back to a depot node while visiting exactly once the other intermediate nodes. The mTSP has been widely studied in the literature. In Cheikhrouhou and Khoufi (2021), Cheikhrouhou and Khoufi present a survey of the problem. They discuss the different variants and approaches as well as the various applications introduced in the literature for the problem. Bektas (2006) presents a survey of the variant where all salesmen are supposed to start from the same depot. In Benavent and Martínez (2013), Benavent and Martínez study the problem where the salesmen are assumed to start from different depots. They give an ILP formulation for the problem and describe several families of valid inequalities along with a substantial polyhedral study. Using this, they devise an efficient Branch-and-Cut algorithm for the problem. Sundar and Rathinam (2016) consider the Generalized multi-depot TSP, a variant that has several applications in ring networks, flexible manufacturing, and scheduling. The authors propose an integer programming formulation for the problem. They also study the associated polytope and derive facet-defining inequalities. These results are used to devise an efficient Branch-and-Cut algorithm. Bernardino et al. (2022) study three variants of the mTSP, the multi-depot family TSP and its two clustered variants. They present several mixed integer linear programming formulations and develop appropriate Branch-and-Cut based algorithms. Cornejo-Acosta et al. (2023) consider the so-called depot-free mTSP where the depots are unknown or unnecessary. New compact integer programming formulations are proposed along with a substantial experimental study. In Duchenne et al. (2007, 2012), Duchenne et al. study the undirected m-Capacitated Peripatetic Salesman Problem. The aim is to determine m Hamiltonian cycles of minimal total cost on a graph, such that all the edges are traversed less than the value of their capacity. In Duchenne et al. (2007), the authors study the polyhedral aspect of the problem, and devise a 2-index Branch-and-Cut algorithm. In Duchenne et al. (2012), they introduce three formulations for the problem, and develop Branch-and-Cut and Branchd-and-Price algorithms.

Another variant, the *Steiner multiple TSP*, closely related to the problem studied in this paper, has been addressed in several works. D'Angelo (2018) considers the Steiner Multi Cycle Problem. Heuristic-based algorithms are described. In Lintzmayer et al. (2020), Lintzmayer et al. study a randomized approximation scheme for the Euclidean Steiner Multi Cycle problem which runs in quasilinear time. Liu et al. (2021) discuss the online Multiple Steiner TSP (mSTSP) with edge blockages. They study the problem with *minsum* and *minmax* objectives. Lower bounds and efficient online algorithms are proposed for both variants.

A further close variant is the *TSP with Precedence Constraints* that consists in finding a minimum-weight tour respecting precedence constraints between some pairs of nodes. This variant has also been widely studied in the literature. In Balas et al. (1995), Balas et al. examine the polytope of the Asymmetric TSP with Precedence Constraints (ATSP-PC). They characterize the dimension of the polytope and derive several families of valid inequalities. They also discuss conditions for these inequalities to be facet-defining and devise separation routines. In Ascheuer et al. (2000), Ascheuer et al. propose a Branch-and-Cut algorithm to solve the ATSP-PC. Gouveia and Pesneau (2006) give extended formulations for the ATSP-PC. Moreover, they describe classes of valid inequalities, and discuss polynomial time separation algorithms. Using this, they devise a cutting plane approach to the problem. The computational results show the efficiency of their approach for reducing the LP gap. In Gouveia et al. (2018), Gouveia et al. propose network-flow-based formulations for the ATSP-PC. Sarin et al. (2014) study variants of the multiple asymmetric TSP with and without precedence constraints and investigate the performances of 32 MILP formulations modeling these variants. Some of these formulations are based on a transformation of the Multiple Asymmetric TSP (mATSP) to Asymmetric TSP (ATSP) (Sarin et al. 2005). In a recent work, Khachai et al. (2023) address the Generalized ATSP-PC. The authors study the associated polytope, identify several families of valid inequalities, and discuss facet-defining conditions. The authors also propose MILP formulations and devise Branch-and-Cut algorithms for the problem.

In what follows, we give notations and definitions that will be used.

1.3 Notations

Let $G = (V, E)$ be a simple edge-weighted undirected graph, such that with each edge $e \in E$ is associated a nonnegative weight w_e. We denote $n = |V|$ and $m = |E|$. An edge e between two nodes u and v in V will be denoted by $e = uv$. We suppose given a set K of salesmen, each having to visit a set of *terminals* $T_k = \{w_1^k, w_2^k, \ldots, w_{|T_k|}^k\}$, $k \in K$. The order of terminals' visitation for each tour (salesman) is assumed to follow the cyclic order of the indices j in w_j^k. Consider a terminal $w_j^k \in T_k$, the terminals in $T_k \setminus \{w_{j-1}^k, w_{j+1}^k\}$ are said to be *non-consecutive*, for w_j^k, $j = 1, 2, \ldots |T_k|$, where $w_{|T_k|+1}^k$ refers to w_1^k. The nodes of V that are not terminals for $k \in K$ are called *Steiner nodes*, and denoted by S_k. If $k \in K$, a pair $q_j^k = (w_j^k, w_{j+1}^k)$ of consecutive terminals w_j^k and w_{j+1}^k is called a *section* for $j \in \{1, 2, \ldots, |T_k|\}$. Each tour, for a salesman $k \in K$, can be seen as the union of node-disjoint sections. We denote by $\mathcal{T}_k$

the set of these sections for $k \in K$. With each section $q_j^k = (w_j^k, w_{j+1}^k) \in \mathcal{T}_k$, $k \in K$ and $j \in \{1, 2, \ldots, |T_k|\}$, we associate a reduced graph, denoted by $G_j^k = (V_j^k, E_j^k)$, obtained from the original graph G by deleting all the terminals of T_k, except w_j^k and w_{j+1}^k, as well as their incident edges. The graph G_j^k will be used to compute a path between w_j^k and w_{j+1}^k not going through any of the other terminals.

Let $V' \subset V$, we denote by $G(V') = (V', E')$ the subgraph induced by the set of vertices V' and the set of edges connecting these vertices $E' = E(V')$. If $W \subset V$ is a subset of nodes of V, we let $\delta_G(W)$ denote the *cut* in G induced by W, that is the set of edges of G having one node in W and the other in $\overline{W} = V \setminus W$. If $W = \{w\}$, $w \in V$, we will write $\delta_G(w)$ for $\delta_G(\{w\})$. In the sequel we simply write $\delta(W)$ and $\delta(w)$, if the context is clear.

1.4 Paper organization

The paper is organized as follows. In the next section, we propose an ILP formulation for the MSTSPOC, and we introduce the associated polytope. In Sect. 3, we study the facial aspect of the basic constraints. In Sect. 4, we describe some valid inequalities and give necessary conditions and sufficient conditions for these inequalities to be facet-defining. In Sect. 5, we give further families of valid inequalities coming from closely related problems. Finally, Sect. 6 is devoted to some concluding remarks. An Appendix is used to give some proofs of facets.

2 Integer linear programming formulation

2.1 ILP formulation

Consider a salesman $k \in K$ and an edge $e \in E$. We define the binary variable x_e^k that is equal to 1 if the tour of salesman k uses edge e, and 0 if not. We also define the binary variable y_e for each $e \in E$, which is equal to 1 if edge e is considered in the final solution, and 0 if not. The MSTSPOC is equivalent to the following ILP.

$$min \sum_{e \in E} w_e y_e \tag{1}$$

$$\sum_{e \in \delta_{G_j^k}(W)} x_e^k \geq 1 \qquad \begin{array}{l} \text{for all } \ k \in K, \ j \in \{1, \ldots, |T_k|\}, \\ q_j^k = (w_j^k, w_{j+1}^k) \in \mathcal{T}_k, \\ W \subset V_j^k : \ w_j^k \in W \text{ and } w_{j+1}^k \in \overline{W}, \end{array} \tag{2}$$

$$\sum_{e \in \delta(w)} x_e^k \leq 2 \qquad \text{for all } \ w \in V, \ k \in K, \tag{3}$$

$$x_e^k \leq y_e \qquad \text{for all } \ e \in E, \ k \in K, \tag{4}$$

$$0 \leq x_e^k \qquad \text{for all } \ e \in E, \ k \in K, \tag{5}$$

$$y_e \leq 1 \qquad \text{for all } \ e \in E, \tag{6}$$

$$x_e^k \in \{0, 1\} \qquad \text{for all } \; e \in E, \; k \in K, \tag{7}$$

$$y_e \in \{0, 1\} \qquad \text{for all } \; e \in E. \tag{8}$$

Inequalities (2) are called *section cut inequalities*. They ensure for each section $q_j^k = (w_j^k, w_{j+1}^k)$, corresponding to a salesman $k \in K$ and $j \in \{1, \dots, |T_k|\}$, a path in the reduced graph G_j^k. Hence, this guarantees, for each salesman a tour going in order through its terminals. Inequalities (3) are called *disjunction inequalities*. They ensure that the different sections for a salesman, $k \in K$ are disjoint, and hence that the associated tour is elementary. Inequalities (4) are the *linking inequalities* which express the fact that if an edge $e \in E$ is not considered in the solution, that is if $y_e = 0$, then e can not be used in any tour for the salesmen K. Finally, inequalities (5) and (6) are the *trivial inequalities*, and (7) and (8) are the *variables' integrality constraints*.

In what follows, we refer to the ILP (1)-(8) as the *cut formulation*. Note that this formulation was first introduced in Borne et al. (2013) and is experimentally studied in Taktak (2024).

2.2 Associated polytope

An instance of the MSTSPOC corresponds to the triplet (G, K, T), where G is a graph, K a set of salesmen, each salesman has a set T_k of terminals and $T = \bigcup\limits_{k \in K} T_k$. We denote by MSTSPOC$(G, K, T)$ the polytope associated with the MSTSPOC, that is the convex hull of the solutions of (2)-(8), *i.e.,*

$$\text{MSTSPOC}(G, K, T) = conv\{(x, y) \in \{0, 1\}^{|E|(|K|+1)} : (x, y) \text{ satisfies (2)-(6)}\}.$$

In what follows, G is assumed to be complete and for each salesman $k \in K$, $T_k \neq V$. These assumptions are not restrictive. In fact, if the graph is not complete, one can consider a complete graph by associating very high costs with the non-existent edges. Moreover, if there exists a salesman $k \in K$ such that $T_k = V$, then the solution is unique for this salesman. In this case, the problem reduces to solving the MSTSPOC for the $K \setminus \{k\}$ remaining salesmen. Furthermore, for convenience, we will suppose that each salesman $k \in K$ has at least 4 Steiner nodes. This will enable us to considerably simplifying some facet proofs.

Give an instance (G, K, T) of MSTSPOC, we denote by $\Omega(G, K, T)$ the set of its solutions. Each solution in $\Omega(G, K, T)$ is represented by the pair (U, I) where $I \subseteq E$ is the set of *installed edges* and $U = (U_1, U_2, ..., U_{|K|}) \subset E^{|K|}$ such that U_j, $j = 1, ..., |K|$, is the set of edges used by salesman j. Given a solution $(U, I) \in \Omega(G, K, T)$, we define (x^U, y^I) its incidence vector. Note that $x^U = (x^{U_1}, x^{U_2}, ..., x^{U_{|K|}})$.

Now, we give a solution for the MSTSPOC that will be used throughout the paper.

Remark 1 Let $U^0 = (U_1^0, U_2^0, ..., U_{|K|}^0)$ be the set of edges between the consecutive terminals of all the salesmen, that is $U_j^0 = \{w_i^j w_{i+1}^j, j \in K, i = 1, ..., |T_j|\}$. And let $I^0 = \bigcup\limits_{j \in K} U_j^0$. Clearly, the pair (U^0, I^0) is a solution of the MSTSPOC.

For sake of presentation, for each of the coming facet proofs, we will begin numbering the solutions (U^j, I^j) by $j = 1$. Having the same name, these solutions may be different from one proof to another.

3 Dimension and facial investigation

In this section, we characterize the dimension of MSTSPOC(G, K, T) and discuss necessary and sufficient conditions for inequalities (2)-(6) to be facet-defining.

3.1 Dimension

In this section, we characterize the dimension of the polytope MSTSPOC(G, K, T). To this end, we first identify the system of its equations.

Remark 2 Consider a salesman $k \in K$ and let $w_j^k \in T_k$ be a terminal. Then w_j^k has a degree equal to 2 in any solution, that is

$$\sum_{e \in \delta(w_j^k)} x_e^k = 2 \qquad \text{for all } w_j^k \in T_k,\ k = 1, \ldots, |K|,\ j \in \{1, \ldots, |T_k|\}. \tag{9}$$

Remark 3 Since equations (9) are written for each terminal $w_j^k \in T_k$, there are $|T_k|$ equations for each salesman $k \in K$.

Remark 4 Consider a salesman $k \in K$, and $w_i^k, w_j^k \in T_k$ be two non-consecutive terminals of k. This yields that edge $e = w_i^k w_j^k$ does not belong to any solution of the MSTSPOC.

$$x_e^k = 0 \qquad \text{for all } e = w_i^k w_j^k,\ w_i^k, w_j^k \in T_k :\ j > i,\ k = 1, \ldots, |K|. \tag{10}$$

Remark 5 Let $G(T_k)$ be the graph induced by the terminals T_k of salesman k. Notice that since G is complete, $G(T_k)$ is also complete and contains $\frac{|T_k|(|T_k|-1)}{2}$ edges. In this graph, there are exactly $|T_k|$ edges between the consecutive terminals. All the remaining edges are between non-consecutive terminals and, in consequence, their number is equal to $\frac{|T_k|(|T_k|-1)}{2} - |T_k| = \frac{|T_k|(|T_k|-3)}{2}$. Consequently, constraints (10) imply $\frac{|T_k|(|T_k|-3)}{2}$ equations of type (10) for salesman k. In the sequel, we shall denote by p_k the number of edges between non-consecutive terminals for salesman k, that is $p_k = \frac{|T_k|(|T_k|-3)}{2}$.

Proposition 1 *Consider an equation $ax + by = \alpha$ of MSTSPOC(G, K, T). Then:*

1. *$b = 0$,*
2. *$ax = \alpha$ is a linear combination of equations (9) and (10).*

Proof We first show that $b = 0$. Consider the feasible solution (U^0, I^0) given by Remark 1. Recall that U_j^0, $j = 1, \ldots, |K|$ consists of the set of edges linking the consecutive terminals of T_j of salesman $j \in K$ and $I^0 = \bigcup_{j \in K} U_j^0$.

Let $e \in E \setminus I^0$ be an arbitrary edge. The solution (U^1, I^1) such that $I^1 = I^0 \cup \{e\}$ and $U^1 = U^0$ also induces a feasible solution of MSTSPOC(G, K, T). This implies that $ax^{U^0} + by^{I^0} = ax^{U^0} + by^{I^0} + b_e$, implying that $b_e = 0$. Since e is arbitrarily chosen in $E \setminus I^0$, we have that

$$b_e = 0 \qquad \text{for all } e \in E \setminus I^0. \tag{11}$$

Now, let us consider the solution (U^2, I^2) obtained as follows. For a salesman $k \in K$, let $e = t_i t_{i+1}$ be an edge between two consecutive terminals t_i and t_{i+1} of T_k. Let $U_k^2 = (U_k^0 \setminus \{e\}) \cup \{t_i s, s t_{i+1}\}$ where s is a Steiner node of salesman k. In addition, let $U_j^2 = U_j^0$, $j = 1, \ldots, K$, $j \neq k$, and $I^2 = \bigcup_{j \in K} U_j^2$. Moreover, consider the solution (U^3, I^3) given by $U^3 = U^2$ and $I^3 = I^2 \cup \{e\}$. Both are feasible solutions for the MSTSPOC and hence their incidence vectors satisfy equation $ax + by = \alpha$. As a consequence, we have $ax^{U^2} + by^{I^2} = ax^{U^2} + by^{I^2} + b_e$, which implies that $b_e = 0$. As e is arbitrary in I^0, this yields

$$b_e = 0 \qquad \text{for all } e \in I^0. \tag{12}$$

By (11) and (12) we then have

$$b_e = 0 \qquad \text{for all } e \in E. \tag{13}$$

Therefore all the equations of MSTSPOC(G, K, T) are given only in terms of variables x.

Let a^k be the restriction of a on salesman $k \in K$. Let $M^k = \begin{pmatrix} M_1^k \\ M_2^k \end{pmatrix}$ be the sub-matrix of equations (9) and (10) involving variables x_e^k, $e \in E$, such that M_1^k is the sub-matrix corresponding to equations (9) and M_2^k the one corresponding to equations (10). To prove that $ax = \alpha$ is a linear combination of equations (9) and (10), it would be sufficient to prove that for each $k \in K$, there exit $\lambda_1^k \in \mathbb{R}^{|T_k|}$ and $\lambda_2^k \in \mathbb{R}^{p_k}$ such that $a^k = \lambda_1^k M_1^k + \lambda_2^k M_2^k$ (recall that p_k is the number of edges between non-consecutive terminals of T_k).

To this end, first we show that for every edge ss' between two Steiner nodes s, s', $a_{ss'}^k = 0$. Consider again the solution (U^0, I^0) and let (U^4, I^4) be the solution given by $U_k^4 = U_k^0 \cup \{ss'\}$, $U_j^4 = U_j^0$ for all $j \in \{1, \ldots, K\} \setminus k$, and $I^4 = \bigcup_{j \in K} U_j^4$. Obviously, (U^4, I^4) is a solution of the MSTSPOC. Since solutions (U^0, I^0) and (U^4, I^4) are both feasible for the MSTSPOC and by (13) $b = 0$, we have that $ax^{U^0} = ax^{U^4} = ax^{U^0} + a_{ss'}^k x_{ss'}^k$. Hence $a_{ss'}^k = 0$, implying that

$$a_{ss'}^k = 0 \qquad \text{for all } s, s' \in S_k, k \in K. \tag{14}$$

Consider now an edge $f = t_i t_{i+1}$ between two consecutive terminals t_i and t_{i+1} of T_k. Let s be a Steiner node of S_k. Consider the solution (U^5, I^5) given by $U_k^5 = (U_k^0 \setminus \{f\}) \cup \{t_i s, s t_{i+1}\}$, $U_j^5 = U_j^0$, for each $j \in \{1, \ldots, K\} \setminus k$, and $I^5 = \bigcup_{j \in K} U_j^5$.

As (U^5, I^5) is feasible for the MSTSPOC, it follows that $ax^{U^0} = ax^{U^5} = ax^{U^0} - a^k_{t_i t_{i+1}} + a^k_{t_i s} + a^k_{s t_{i+1}}$, implying that $a^k_{t_i t_{i+1}} = a^k_{t_i s} + a^k_{s t_{i+1}}$. Therefore, as nodes s, t_i and t_{i+1} are all arbitrary, we have

$$a^k_{t_i t_{i+1}} = a^k_{t_i s} + a^k_{s t_{i+1}} \qquad \text{for all } t_i, t_{i+1} \in T_k,\ s \in S_k, k \in K. \tag{15}$$

Now, we will prove that all the edges linking a terminal to Steiner nodes of S_k have the same coefficient a^k_e. Consider (U^6, I^6) obtained from (U^5, I^5) as follows: $U^6_k = (U^5_k \backslash \{st_i\}) \cup \{ss', s't_i\}$, $U^6_j = U^5_j$ for each $j \in \{1, \ldots, K\} \backslash k$ and $I^6 = \bigcup_{j \in K} U^6_j$, where s and s' are Steiner nodes of S_k (Recall that we assumed that $|S_k| \geq 2$ for all $k \in K$). Since (U^5, I^5) and (U^6, I^6) satisfy equation $ax + by = \alpha$, this implies that $ax^{U^5} = ax^{U^6} = ax^{U^5} - a^k_{st_i} + a^k_{ss'} + a^k_{s't^k_i}$, and hence $a^k_{st_i} = a^k_{ss'} + a^k_{s't_i}$. By (14), it follows that

$$a^k_{st_i} = a^k_{s't_i} = \lambda^k_1(t_i) \quad \begin{array}{l} \text{for all } t_i \in T_k,\ s, s' \in S_k, k \in K, \\ \text{for some } \lambda^k_1(t_i) \in \mathbb{R}. \end{array} \tag{16}$$

Now, let $\lambda^k = (\lambda^k_1, \lambda^k_2)$, $k \in K$ such that $\lambda^k_1 = (\lambda^k_1(t_i), t_i \in T_k)$ and $\lambda^k_2 = (\lambda^k_2(uv),\ u, v \in T_k, uv \notin U^0_k)$ such that $\lambda^k_2(uv) = a^k_{uv} - \lambda^k_1(u) - \lambda^k_1(v)$, $k \in K$. By (15), we have $a^k_{t_i t_{i+1}} = \lambda^k_1(t_i) + \lambda^k_1(t_{i+1})$ for each two consecutive terminals t_i and t_{i+1} of T_k.

The vectors λ^k_1 and λ^k_2 can then be given so that:

$$a^k_{uv} = \begin{cases} \lambda^k_1(u) + \lambda^k_1(v) & \text{if } uv = t_i t_{i+1},\ t_i, t_{i+1} \in T_k, \\ \lambda^k_1(u) & \text{if } u \in T_k,\ v \in S_k, \\ \lambda^k_2(uv) + \lambda^k_1(u) + \lambda^k_1(v) & \text{if } uv = t_i t_j,\ t_i, t_j \in T_k :\ j > i, \\ 0 & \text{if } u, v \in S_k,\ u \neq v, \end{cases}$$

yielding

$$a^k = \lambda^k_1 M^k_1 + \lambda^k_2 M^k_2 \qquad \text{for all } k \in K,$$

as desired. □

By Proposition 1, we know that the only equations of MSTSPOC(G, K, T) are equations (9) and (10). The matrix M of equations of MSTSPOC(G, K, T) can hence be written as

$$M = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_{|K|} \end{pmatrix}$$

where M_k is the matrix of the equations system (9) and (10) involving variables x^k_e for salesman $k \in K$. Each matrix M_k, $k = 1, \ldots, |K|$, consists of two sub-matrices.

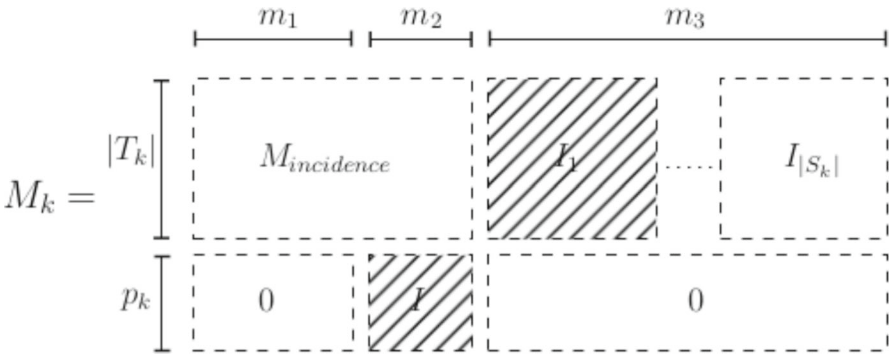

Fig. 2 Matrix of equations for salesman k

The first one is a terminal-edge incidence matrix and the second is a restriction of the identity matrix for the edges between non-consecutive terminals. More precisely, each matrix M_k can be organized as in Fig. 2. The first m_1 columns correspond to the edges between consecutive terminals ($m_1 = |T_k|$). The next m_2 columns correspond to the edges between non-consecutive terminals ($m_2 = p_k$). The last m_3 columns correspond to the edges between the terminals of T_k and the Steiner nodes of S_k for $k = 1, \dots, |K|$. Here, the first columns are related to the Steiner node s_1, the second to s_2, and so on. The first $|T_k|$ rows of M_k are associated with the terminals of T_k, and the last ones are associated with the edges between non-consecutive terminals. Note that the first $|T_k|$ rows of M_k correspond to equations (9) and the last p_k ones correspond to equations (10). The sub-matrices $I_1, \dots, I_{|S_k|}$ are identity matrices and are related to the Steiner nodes $s_1, \dots, s_{|S_k|}$ of S_k, respectively.

Observe that M_k contains two blocks consisting of two identity matrices, namely I_1 and I (hatched matrices). Moreover, these matrices cover the rows of M_k. Hence M_k is of full rank equal to $|T_k| + p_k = \frac{|T_k|(|T_k|-1)}{2}$. Therefore

$rank(M) = \sum_{k \in K} rank(M_k) = \sum_{k \in K} \frac{|T_k|(|T_k|-1)}{2}$.

As a consequence, we have the following result.

Theorem 1

$$dim(MSTSPOC(G, K, T)) = (|K| + 1)|E| - \sum_{k \in K} \frac{|T_k|(|T_k| - 1)}{2}.$$

3.2 Facial investigation

In this section, we study the facial structure of the polytope MSTSPOC(G, K, T). In particular, we give necessary and sufficient conditions for inequalities of the cut formulation to be facet defining.

For convenience, we will suppose that each salesman has at least 4 Steiner nodes.

Theorem 2 *Inequality $x_e^k \geq 0$ defines a facet of $MSTSPOC(G, K, T)$ if and only if e is not between non-consecutive terminals of T_k.*

Proof Let

$$F_e^k = \{(x, y) \in \mathrm{MSTSPOC}(G, K, T) : x_e^k = 0\}.$$

The necessity condition is a consequence of Remark 4.

In what follows, we suppose that e is not an edge between non-consecutive terminals of T^k.

Denote inequality $x_e^k \geq 0$ by $ax + by \leq \alpha$ and let $rx + qy \leq \beta$ be a valid inequality defining a facet F of $\mathrm{MSTSPOC}(G, K, T)$. Assume that $F_e^k \subseteq F$. To prove that F_e^k is a facet of $\mathrm{MSTSPOC}(G, K, T)$, it suffices to show that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^j, j \in K)$, $\lambda^j \in \mathbb{R}^{|T_j|+p_j}$ for $j \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$ (where $r = (r^1, r^2, \ldots, r^{|K|})$ with $r^j \in \mathbb{R}^m$, $j = 1, \ldots, |K|$ and M is the equations matrix defined above). Note here that $a = (a^1, a^2, \ldots, a^{|K|})$ is such that $a^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ with $a^i = 0$ for $i \in \{1, \ldots, |K|\} \setminus \{k\}$, $a_e^k = 1$ and $a_{e'}^k = 0$ for $e \neq e'$. Note also that $b = 0$.

We will consider the case $e = s_1 s_2$, where s_1 and s_2 are Steiner nodes of S_k, the proof for the remaining cases is along the same line.

First, we prove that $q = 0$.

Consider the solution (U^0, I^0) given in Remark 1 and let $f \in E \setminus I^0$. The solution (U^1, I^1) such that $I^1 = I^0 \cup \{e\}$ and $U^1 = U^0$ also induces a feasible solution of $\mathrm{MSTSPOC}(G, K, T)$. Moreover, the incidence vectors of these solutions are in F_e^k and hence in F. This implies that $rx^{U^0} + qy^{I^0} = rx^{U^0} + qy^{I^0} + q_e$, and hence $q_f = 0$. Since f is arbitrarily chosen in $E \setminus I^0$, we obtain that

$$q_f = 0 \qquad \text{for all } f \in E \setminus I^0. \tag{17}$$

Now, consider the solution (U^2, I^2) obtained as follows. Consider a salesman $l \in K$ and let $f = t_i t_{i+1}$ be an edge between two consecutive terminals t_i and t_{i+1} of T_l. Let $U_l^2 = (U_l^0 \setminus \{e\}) \cup \{t_i s, s t_{i+1}\}$ where $s \in S_l$. In addition, let $U_j^2 = U_j^0$, $j = 1, \ldots, |K|$, $j \neq l$, and $I^2 = \bigcup_{j \in K} U_j^2$. And, let (U^3, I^3) be the solution such that $U^3 = U^2$ and $I^3 = I^2 \cup \{e\}$. Clearly, both solutions are feasible for the MSTSPOC and are in F_e^k. Hence, $rx^{U^2} + qy^{I^2} = rx^{U^2} + qy^{I^2} + q_e$, which implies that $q_f = 0$. As f is arbitrary in I^0, this yields

$$q_f = 0 \qquad \text{for all } f \in I^0. \tag{18}$$

By (17) and (18), we then have

$$q_f = 0 \qquad \text{for all } f \in E. \tag{19}$$

We now show that $r_{ss'}^j = 0$ for all $s, s' \in S_j$ such that either $j = k$ and $ss' \neq e$ or $j \in K \setminus \{k\}$.

Consider an edge ss' between two Steiner nodes s and s' of S_k different from e. Let (U^4, I^4) be the solution defined as follows: $U_k^4 = U_k^0 \cup \{ss'\}$, $U_j^4 = U_j^0$ for all $j \in \{1, \ldots, |K|\} \setminus \{k\}$ and $I^4 = \bigcup_{j \in K} U_j^4$. As $ss' \notin U_k^0$ and $ss' \notin U_k^4$,

$(x^{U^0}, y^{I^0}), (x^{U^4}, y^{I^4}) \in F_e^k$, and consequently $(x^{U^0}, y^{I^0}), (x^{U^4}, y^{I^4}) \in F$. By (19), it follows that $rx^{U^0} = rx^{U^4}$. Therefore $rx^{U^0} = rx^{U^0} + r^k_{ss'}$, and hence $r^k_{ss'} = 0$. As s and s' are arbitrary in S_k, we then obtain that

$$r^k_{ss'} = 0 \qquad \text{for all } \; s, s' \in S_k, ss' \neq e. \tag{20}$$

Similarly we can show that

$$r^j_{ss'} = 0 \qquad \text{for all}\, ss' \in S_j, l \in K \setminus \{k\}. \tag{21}$$

Now, we will consider edges between consecutive terminals.

Consider a salesman $l \in K$ and let $t_i t_{i+1}$ be an edge between two consecutive terminals t_i and t_{i+1} of T_l. Consider a Steiner node s of S_l and let (U^6, I^6) be the solution given by $U_l^6 = (U_l^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, s t_{i+1}\}$, $U_j^6 = U_j^0$ for each $j \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^6 = \bigcup_{j \in |K|} U_j^6$. As (x^{U^6}, y^{I^6}) is in F_e^k and hence in F, by (19) we have $rx^{U^0} = rx^{U^0} - r^l_{t_i t_{i+1}} + r^l_{t_i s} + r^l_{s t_{i+1}}$ consequently, $r^l_{t_i t_{i+1}} = r^l_{t_i s} + r^l_{s t_{i+1}}$. As l, t_i, t_{i+1} and s are arbitrarily chosen, we have

$$r^l_{t_i t_{i+1}} = r^l_{t_i s} + r^l_{s t_{i+1}} \quad \text{for all } \; t_i, t_{i+1} \in T_l, \; s \in S_l, l \in K. \tag{22}$$

Now, we will prove that all the edges linking a terminal to Steiner nodes have the same coefficient.

Consider a salesman $l \in K \setminus \{k\}$ and two Steiner nodes s and s' in S_l. Denote (U^7, I^7) the solution given by $U_l^7 = (U_l^6 \backslash \{s t_i\}) \cup \{ss', s' t_i\}$, $U_j^7 = U_j^6$ for each $j \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^6 = \bigcup_{j \in K} U_j^6$. Clearly, (x^{U^7}, y^{I^7}) is in F_e^k and thus in F. The incidence vectors of (U^6, I^6) and (U^7, I^7) satisfy equation $rx + qy = \beta$. As $q = 0$, this implies that $rx^{U^6} = rx^{U^7} = rx^{U^6} - r^l_{st_i} + r^l_{ss'} + r^l_{s't_i}$. Hence $r^l_{st_i} = r^l_{ss'} + r^l_{s't_i}$. By (21), we obtain that $r^l_{st_{i+1}} = r^l_{s't_{i+1}}$. As salesman l, and nodes s, s' and t_{i+1} are arbitrary, we then have

$$r^l_{st_i} = r^l_{s't_i} \qquad \text{for all } \; t_i \in T_l, \; s, s' \in S_l, l \in K \setminus \{k\}.$$

In a similar way, we can show that

$$r^k_{st_i} = r^k_{s't_i} \qquad \text{for all } \; t_i \in T_k, \; s, s' \in S_k, \; ss' \neq e.$$

The previous relation can be extended to s_1, s_2. In fact, if $\overline{s} \in S_k \setminus \{s_1, s_2\}$, by the relations above for every terminal t_i of T_k, $r^k_{t_i s_1} = r^k_{t_i \overline{s}}$ and $r^k_{t_i s_2} = r^k_{t_i \overline{s}}$. As a consequence, we have $r^k_{t_i s_1} = r^k_{t_i s_2}$.

Overall, we obtain that

$$r^l_{st_i} = r^l_{s't_i} = \lambda_1^l(t_i) \quad \begin{array}{l} \text{for all } \; t_i \in T_l, \; s, s' \in S_l, l \in K, \\ \text{for some } \lambda_1^l(t_i) \in \mathbb{R}. \end{array} \tag{23}$$

Now, let $\rho = r^k_{s_1 s_2}$ and $\lambda^l = (\lambda^l_1, \lambda^l_2)$, $l \in K$ such that $\lambda^l_1 = (\lambda^l_1(t_i), t_i \in T_k)$ where $\lambda^l_1(t_i)$ is as given by (23) and $\lambda^l_2 = (\lambda^l_2(uv),\ u, v \in T_k, uv \notin U^0_l)$ such that $\lambda^l_2(uv) = r^l_{uv} - \lambda^l_1(u) - \lambda^l_1(v), l \in K$.

Overall, we obtain that

$$r^l_{uv} = \begin{cases} \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_i t_{i+1},\ t_i, t_{i+1} \in T_l, \\ \lambda^l_1(u) & \text{if } u \in T_l, \\ \lambda^l_2(uv) + \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_i t_j,\ t_i, t_j \in T_l,\ j > i, \\ 0 & \text{if } l \neq k \text{ and } u, v \in S_l,\ u \neq v, \\ 0 & \text{if } l = k \text{ and } u, v \in S_k,\ uv \neq s_1 s_2, \\ \rho & \text{if } l = k \text{ and } uv = s_1 s_2. \end{cases}$$

It is not hard to see that, $r^l = \rho a^l + \lambda^l_1 M^l_1 + \lambda^l_2 M^l_2$, for all $l \in K$. Consequently, $r = \rho a + \lambda M$. □

Theorem 3 *Inequality $y_e \leq 1$ defines a facet of MSTSPOC(G, K, T).*

Proof See Appendix A, Sect. A.1. □

In what follows, we study the facial structure of the section cut inequalities (2). Consider a salesman $k \in K$ and a section $q^k_j = (t_j, t_{j+1}) \in \mathcal{T}_q$. Consider W a subset of nodes of V^k_j such that $t_j \in W$ and $t_{j+1} \in \overline{W}$.

Theorem 4 *Inequality (2) defines a facet of MSTSPOC(G, K, T) if and only if $W \cap S_k \neq \emptyset \neq \overline{W} \cap S_k$.*

Proof See Appendix A, Sect. A.2. □

Now, we examine the facial structure of the disjunction inequalities (3).

Theorem 5 *Consider a salesman $k \in K$, and $w \in V$.*
Inequality $\sum\limits_{e \in \delta(w)} x^k_e \leq 2$ defines a facet for MSTSPOC(G, K, T) if and only if w is not a terminal of T_k.

Proof Let F^k_w define the face induced by inequality $\sum\limits_{e \in \delta(w)} x^k_e \leq 2$, that is

$$F^k_w = \{(x, y) \in \text{MSTSPOC}(G, K, T) : \sum_{e \in \delta_G(w)} x^k_e = 2\}.$$

The necessity follows from Remark 2.

Suppose that $w \in S_k$, that is to say w is a Steiner node of salesman k. Let $T_k = \{t_1, t_2, t_3, \ldots, t_{|T_k|}\}$. We prove that inequalities (3) are facet defining by exhibiting dim(MSTSPOC(G, K, T)) points in F^k_w that are affinely independent. These will be determined in two steps.

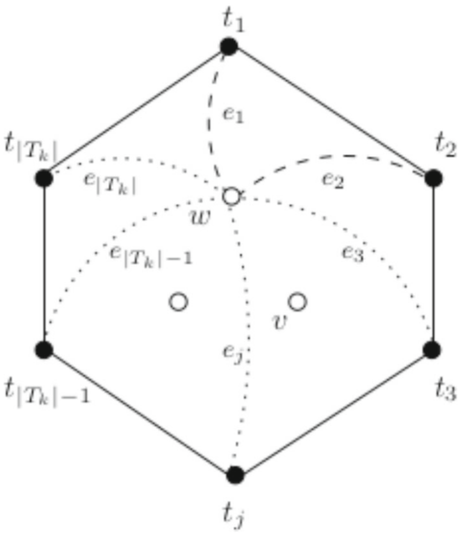

Fig. 3 Facial aspect of disjunction inequalities: configuration

We first exhibit $q_1 = (|K| + 1)|E| - \sum_{j \in K} \frac{|T_j|(|T_j|-1)}{2} - |T_k| + 1$ solutions of MSTSPOC(G, K, T) that are in F_w^k. These can be obtained as follows. Consider, without loss of generality, the first section of salesman k, that is $(t_1, t_2) \in \mathcal{T}_k$ (see Fig. 3). Figure 3 shows the edges $e_1, e_2, \ldots, e_{|T_k|-1}, e_{|T_k|}$. These are the edges linking the different terminals of T_k to the node w. Now, consider all the possible solutions of MSTSPOC(G, K, T) that are obtained by inserting the Steiner node w between t_1 and t_2. Note that these solutions all use edges e_1 and e_2. Note also that, since w always appears in these solutions, w can be considered as a terminal for salesman k. Hence, the incidence vectors of these solutions are all in F_w^k and are nothing but the possible solutions of the problem when w is added as terminal of T_k between t_1 and t_2. Note that all these solutions do not use any of the edges $e_3, e_4, \ldots, e_{|T_k|-1}, e_{|T_k|}$. Hence, all the solutions built this way are also solutions of the polytope MSTSPOC(G, K, T'), where $T' = (T \backslash T_k) \cup T_k^{'}$ with $T_k^{'} = T_k \cup \{w\}$, is the new set of terminals of salesman k.

By Theorem 1, there are $dim($MSTSPOC$(G, K, T')) + 1$ solutions affinely independent in MSTSPOC(G, K, T'). As all these solutions are in F_w^k and $|T_k'| = |T_k| + 1$, we then obtain $(|K| + 1)|E| - \sum_{j \in K} \frac{|T_j|(|T_j|-1)}{2} - |T_k| + 1$ solutions affinely independent in F_w^k.

Now, we will exhibit $q_2 = |T_k| - 1$ further solutions in F_w^k.

Consider a Steiner node $v \in S_k \backslash \{w\}$ and let (U^j, I^j), $j = 2, \ldots, |T_k|$, be the solution given by $U_k^j = (U_k^0 \backslash \{t_j t_{j+1}\}) \cup \{t_j w, wv, vt_{j+1}\}$, $U_p^j = U_p^0$ for each $p \in K \backslash \{k\}$, and $I^j = \bigcup_{p \in K} U_p^j$. Observe that solution (U^j, I^j) is nothing but (U^0, I^0) for which we delete edge $t_j t_{j+1}$ and insert the Steiner nodes w and v respectively, between t_j and t_{j+1}. Also note that exactly two edges related to salesman k are incident to w in each of these solutions. Hence, the incidence vectors of these solutions all belong to F_w^k. Furthermore, it is clear that their incidence vectors are affinely independent.

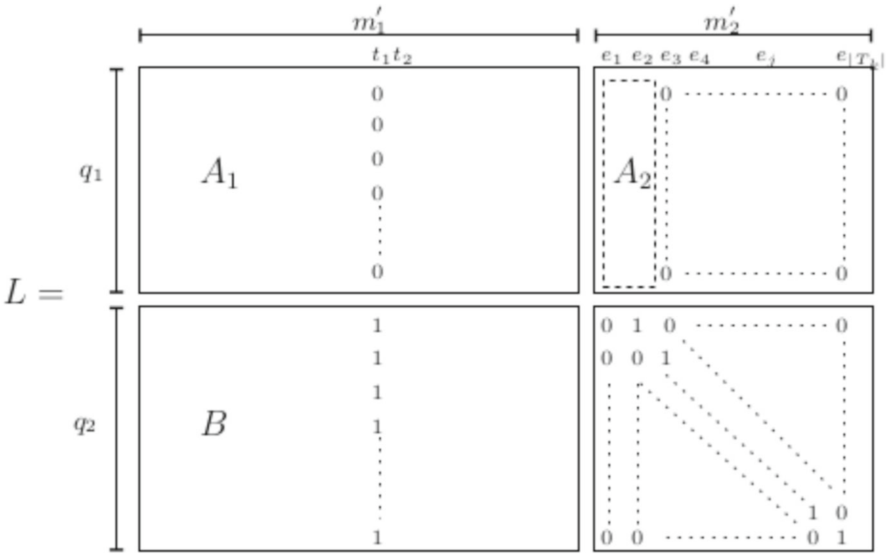

Fig. 4 Facial aspect of disjunction inequalities: matrix

Let L be the matrix whose rows are the incidence vectors of all the solutions obtained above (see Fig. 4). Matrix L is organized as follows. The first m_1' columns of L correspond to all the variables $y_e, e \in E$ and $x_e^l, e \in E, l \in K$ except the variables $x_{e_1}^k, x_{e_2}^k, \ldots, x_{e_{|T_k|}}^k$. The last m_2' columns of L are associated with the variables $x_{e_1}^k, x_{e_2}^k, \ldots, x_{e_{|T_k|}}^k$. The first q_1 rows of L correspond to the solutions obtained in the first step of the proof. The ones built in the second step correspond to the last q_2 rows of L.

Observe that the first q_1 solutions use only edges among e_1 and e_2 and do not use the $m_2'-2$ last edges of matrix L, that are edges between w and terminals $t_3, \ldots, t_{|T_k|}$. Moreover, these solutions do not use the edge t_1t_2. The other q_2 solutions uses each time only one of the m_2' last edges that is different from e_1. Remark also that, in contrast with the first q_1 solutions, all the last p_2 solutions use the edge t_1t_2. As the first q_1 solutions are affinely independent, it can be easily seen that the $q_1 + q_2$ solutions are affinely independent.

Thus, we obtain

$q_1+q_2 = (|K|+1)|E| - \sum_{j\in K} \frac{|T_j|(|T_j|-1)}{2} = dim(\text{MSTSPOC}(G, K, T))$ affinely independent solutions of F_w^k, which ends the proof. □

4 Valid and facet-defining inequalities

In this section, we present several families of valid inequalities for the MSTSPOC. The main objective of this section is to study the facial aspects of these inequalities. In particular, we describe necessary conditions and sufficient conditions for these inequalities to be facet-defining. For the validity of these inequalities the reader is referred to Taktak (2024).

4.1 Steiner cut inequalities

The first family of valid inequalities is a straight consequence related to the connectivity requirements of the problem. For a salesman $k \in K$, as the solution is a Steiner tour, every cut separating terminals of T_k must contain at least 2 edges (Taktak 2024).

Proposition 2 *Consider a salesman $k \in K$ and let $W \subset V$ such that $W \cap T_k \neq \emptyset \neq \overline{W} \cap T_k$. Then*

$$\sum_{e \in \delta(W)} x_e^k \geq 2 \tag{24}$$

is valid for MSTSPOC(G, K, T).

Inequalities of type (24) are called *Steiner cut inequalities*.

Theorem 6 *Inequality (24) defines a facet of MSTSPOC(G, K, T) if and only if the following hold.*

1. *W and $\overline{W}$ do not contain non-consecutive terminals of T_k, with at least one of them having its successor and predecessor in $\overline{W}$.*
2. *If $|W \cap T_k| = 2$ (resp. $|\overline{W} \cap T_k| = 2$), then $W \cap S_k \neq \emptyset$ (resp. $\overline{W} \cap S_k \neq \emptyset$),*
3. *If $|W \cap T_k| \geq 3$ (resp. $|\overline{W} \cap T_k| \geq 3$), then $S_k \subset W$ (resp. $S_k \subset \overline{W}$).*

Proof Let F_W^k be the face induced by inequality (24) that is

$$F_W^k = \{(x, y) \in \text{MSTSPOC}(G, K, T) : \sum_{e \in \delta(W)} x_e^k = 2\}.$$

Neccessity.

(1) Suppose that W contains non-consecutive terminals t_i and t_j (see Fig. 5). Consider a solution (U, I) of MSTSPOC. By (2), t_i must be linked to t_{i-1} and t_{i+1} by two disjoint paths. Denote by $P_{i,1}$ and $P_{i,2}$ these paths. Also, there must exist a path P_j linking t_j to $\overline{W}$. As t_i and t_j are non-consecutive, this implies that the paths $P_{i,1}$, $P_{i,2}$, and P_j all intersect cut $\delta(W)$. Thus, $x^k(\delta(W)) \geq 3$ and therefore (x^U, y^I) does not belong to F_W^k. But this implies that $F_W^k = \emptyset$, and hence it can not define a facet.

(2) Now, suppose that condition 1) is satisfied, however condition 2) is not. Suppose for instance that $|W \cap T_k| \geq 3$ and $\overline{W} \cap S_k \neq \emptyset$. Consider, without loss of generality, the case where $|W \cap T_k| = 3$. Denote the terminals of T_k in W by t_1, t_2 and t_3 (see Fig. 6).

Consider a solution (U, I) of MSTSPOC, with $U = (U_1, U_2, \ldots, U_{|K|})$, such that $(x^U, y^I) \in F_W^k$. Let $e = t_2 s$ be the edge between t_2 and a Steiner node $s \in \overline{W}$ (see Fig. 6). Suppose that $e \in U_k$. From constraints (2), there exist a path P_1 between t_1 and its successor in $\overline{W}$, a path P_3 between t_3 and its predecessor in $\overline{W}$ and a path P_4 between

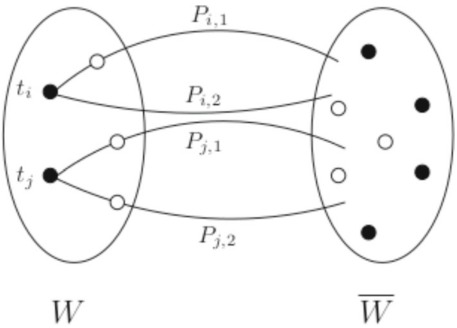

Fig. 5 Facial aspect for Steiner Cut inequalities: condition 1

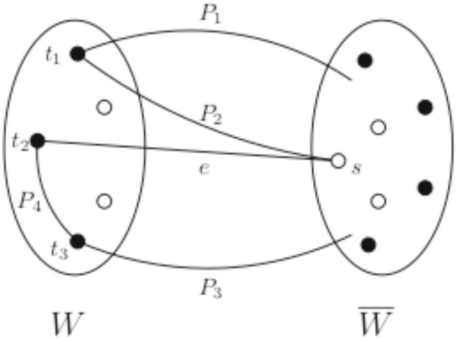

Fig. 6 Facial aspect for Steiner Cut inequalities: condition 2

t_2 and t_3. Moreover, as $e \in U_k$, by (2), it follows that e must belong to the path joining t_1 and t_2. Therefore, there must exist a path between the Steiner node s and the terminal t_1. This path is denoted P_2. Notice that P_1, P_2, P_3 and $\{e\}$ all intersect cut $\delta(W)$ (see Fig. 6). As a consequence, $x^k(\delta(W)) \geq 4$ and hence (x^U, y^I) does not belong to F_W^k. This implies that every solution of MSTSPOC such that $(x^U, y^I) \in F_W^k$ satisfies $x_e^k = 0$. Hence $F_W^k \subset F_e^k$, where $F_e^k = \{(x, y) \in \text{MSTSPOC}(G, K, T) : x_e^k = 0\}$. Here, e is not between non-consecutive terminals. By Theorem (2), it follows that F_e^k define a facet of MSTSPOC(G, K, T). Moreover, inequality (24) cannot be obtained as a combination of $x_e^k \geq 0$ and the equations of MSTSPOC(G, K, T), i.e. 9 and 10. Consequently, F_W^k cannot define a facet of MSTSPOC(G, K, T).

3) Now, we will suppose that conditions 1) and 2) are satisfied. Assume however that condition 3) is not satisfied. Suppose, for example, that $|W \cap T_k| = 2$ but $W \cap S_k = \emptyset$ (see Fig. 7).

Consider a solution (U, I) of MSTSPOC with $U = (U_1, U_2, \ldots, U_{|K|})$ such that $t_1t_2 \notin U_k$. As $x^k(t_1) = 2$ and $x^k(t_2) = 2$ hold, it follows that $x^k(\delta(W)) = 4$, and hence $(x^U, y^I) \notin F_W^k$. But this implies that F_W^k is contained in the face induced by $x_{t_1t_2}^k \leq 1$. Hence F_W^k cannot define a facet of MSTSPOC(G, K, T).

Sufficiency.

See Appendix B, Sect. B.1. □

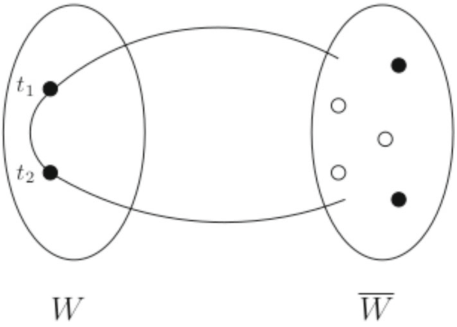

Fig. 7 Facial aspect for Steiner Cut inequalities: condition 3

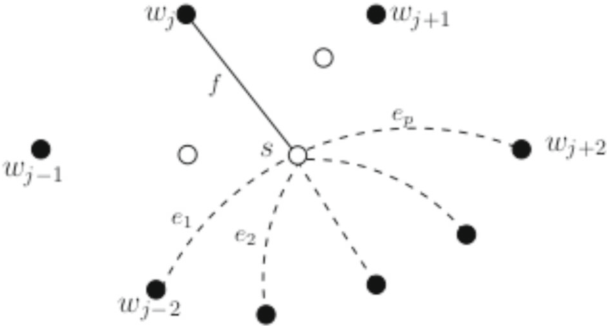

Fig. 8 Steiner non-consecutive terminals: configuration 1

4.2 Steiner non-consecutive terminals inequalities

In this section, we introduce a further family of valid inequalities for the MSTSPOC.

Consider a salesman $k \in K$ and let w_j be a terminal of T_k. Consider a Steiner node s of S_k and denote $f = sw_j$. Denote the edges linking the Steiner node s with the terminals of T_k non-consecutive to w_j by $e_1, e_2, \dots, e_p$ (see Fig. 8).

Remark that if the edge f is considered in a solution $\mathcal{S}$, it can be used to route only one among the sections (w_{j-1}, w_j) and (w_j, w_{j+1}). Thus, (exactly) one edge of $\delta'(s) = \delta(s)\backslash\{f, e_1, e_2, \dots, e_p\}$ will be used in the solution $\mathcal{S}$, and none of the edges $e_1, e_2, \dots, e_p$ could be considered in $\mathcal{S}$. Therefore the inequality

$$\sum_{e \in \delta'(s)} x_e^k \geq x_f^k, \tag{25}$$

is valid.

In the following, we propose a generalization of inequality (25). Consider a salesman $k \in K$ such that $|T_k| \geq 4$ and let w_j be a terminal of T_k. Consider a subset of Steiner nodes $S \subset S_k$ and let $\Pi = (V_0, V_1, \dots, V_p)$, $p \geq 4$ be a partition of V (see Fig. 9) such that:

1. $V_0 = S$,
2. $V_1 \cap T_k = \{w_{j-l}, \dots, w_{j-2}, w_{j-1}\}$,

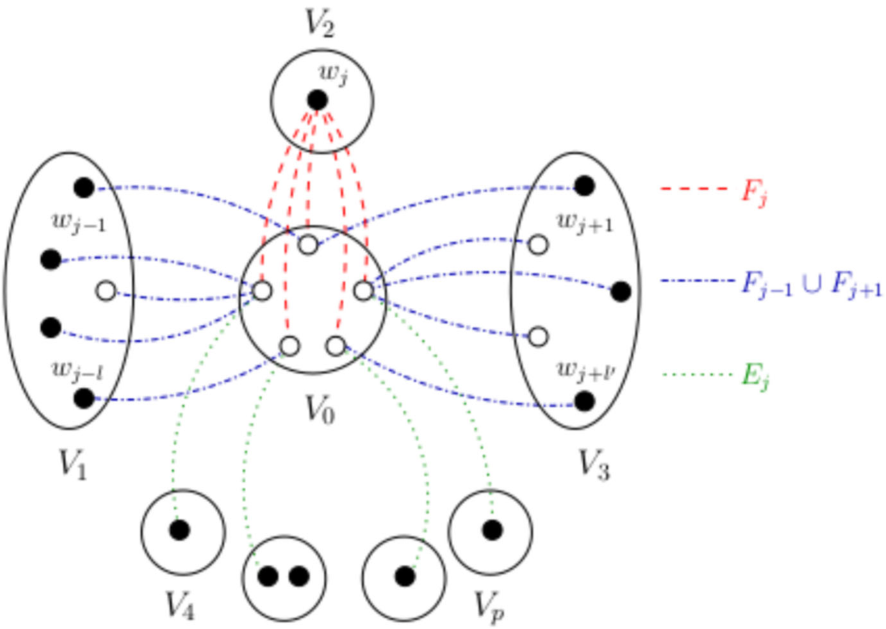

Fig. 9 Steiner non-consecutive terminals: configuration 2

3. $V_2 = \{w_j\}$,
4. $V_3 \cap T_k = \{w_{j+1}, w_{j+2}, \ldots, w_{j+l'}\}$,
5. $V_4, \ldots, V_p$ are such that $V_i \cap T_k \neq \emptyset$ and $V_i \cap S_k = \emptyset$, $i = 4, \ldots, p$ (see Fig. 9).

Denote by F_{j-1}, F_j, F_{j+1} and E_j the sets of edges of E such that $F_{j-1} = [V_0, V_1]$, $F_j = [V_0, V_2]$, $F_{j+1} = [V_0, V_3]$, and $E_j = \bigcup_{i=4}^{p}([V_0, V_i])$. With partition Π and the sets F_{j-1}, F_j, F_{j+1} and E_j, we associate the following inequality

$$\sum_{e \in \delta'(S)} x_e^k \geq \sum_{e \in F_j} x_e^k , \tag{26}$$

where $\delta'(S) = F_{j-1} \cup F_{j+1} = \delta(S) \backslash \{E_j, F_j\}$. Inequality (26) express the fact that the flow going from w_j to a subset of Steiner nodes $S \subseteq S_k$ must be conserved in S and only used to route sections that are adjacent to w_j.

Proposition 3 *Inequality (26) is valid for MSTSPOC(G, K, T).*

Proof See Taktak (2024). □

Inequalities of type (26) will be called *Steiner non-consecutive terminals inequalities*.

Theorem 7 *Inequality (26) defines a facet of MSTSPOC(G, K, T) if and only if $V_1 \cap T_k = \{w_{j-1}\}$ and $V_3 \cap T_k = \{w_{j+1}\}$.*

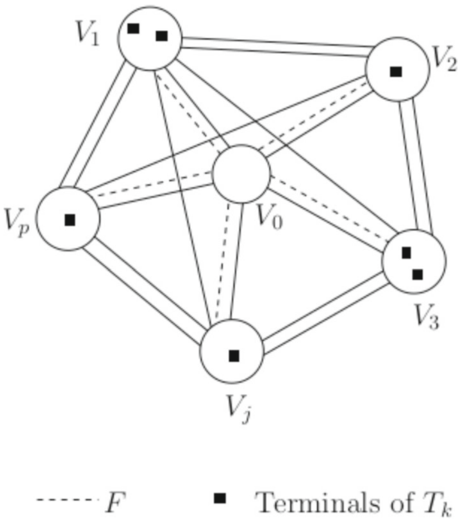

Fig. 10 Steiner F-partition configuration

Proof See Appendix B, Sect. B.2. □

4.3 *F*-partition inequalities

The F-partition inequalities were first introduced by Mahjoub (1994). Further works have shown the efficiency of this class of inequalities in solving different variants of the survivable network design problem (see for instance Bendali et al. 2010; Huygens et al. 2004; Mahjoub and Pesneau 2008). In what follows, we discuss a variant of these inequalities for the MSTSPOC.

Proposition 4 *Consider a salesman $k \in K$ and let $\Pi = (V_0, \ldots, V_p)$, $p \geq 2$ be a partition of V such that $|V_i \cap T_k| \geq 1, i = 1, \ldots, p$. Let $F \subseteq \delta(V_0)$ such that $|F|$ is odd. Then*

$$x^k(\delta(V_0, \ldots, V_p) \setminus F) \geq p - \left\lfloor \frac{|F|}{2} \right\rfloor \tag{27}$$

is valid for MSTSPOC(G, K, T), and is called Steiner F-partition inequality.

Proof The following inequalities are valid for MSTSPOC(G, K, T)

$$\begin{array}{lll} x^k(\delta(V_i)) & \geq 2 & \text{for all } \; i = 1, \ldots, p, \\ -x^k(f) & \geq -1 & \text{for all } \; f \in F, \\ x^k(g) & \geq 0 & \text{for all } \; g \in \delta(V_0) \setminus F. \end{array}$$

By summing these inequalities, we obtain

$$2x^k(\delta(V_0, \ldots, V_p) \setminus F) \geq 2p - |F|$$

By dividing by 2 and rounding up the right-hand side, we obtain

$$x^k(\delta(V_0,\ldots,V_p)\setminus F) \geq p - \lfloor \frac{|F|}{2} \rfloor.$$

□

Observe that if $|F| = 2q+1$ for some integer q, inequality (27) can also be written as

$$x^k(\delta(V_0,\ldots,V_p)\setminus F) \geq p - q. \tag{28}$$

Now, we will study the facial aspect of these inequalities.

First, we give necessary conditions for these inequalities to be facet-defining.

Theorem 8 *Inequality (27) defines a facet for MSTSPOC(G, K, T) only if*

1. *Each V_i, $i \in \{1,\ldots,p\}$ is such that*
    - *(i) V_i does not contain non-consecutive terminals of T_k,*
    - *(ii) if V_i contains (at least) three consecutive terminals, then $S_k \subset V_i$,*
    - *(iii) if V_i contains exactly two consecutive terminals, then $V_i \cap S_k \neq \emptyset$.*
2. *F does not contain an edge between non-consecutive terminals of T_k,*
3. *if s and s' are two Steiner nodes of S_k such that $s \in V_i$ and $s' \in V_j$, $i, j \in \{1,\ldots,p\}$ and $i \neq j$, then V_i and V_j contain consecutive terminals.*

Proof Let $F^k_{\Pi,F}$ be the face induced by inequality (27), that is

$$F^k_{\Pi,F} = \{(x,y) \in \text{MSTSPOC}(G,K,T) : x^k(\delta(V_0,\ldots,V_p)\setminus F) = p - \lfloor \frac{|F|}{2} \rfloor\},$$

Necessity

1) Suppose that condition 1) is not satisfied.

i) Suppose that V_i contains non-consecutive terminals of T_k. Then, the following hold for any solution of MSTSPOC(G, K, T)

$$\begin{array}{lll} x^k(\delta(V_i)) & \geq 4 & \\ x^k(\delta(V_j)) & \geq 2 & \text{for all } \; j \in \{1,\ldots,p\}\setminus\{i\}, \\ -x^k(f) & \geq -1 & \text{for all } \; f \in F, \\ x^k(g) & \geq 0 & \text{for all } \; g \in \delta(W)\setminus F. \end{array}$$

By summing these inequalities, we obtain

$$2x^k(\delta(V_0,\ldots,V_p)\setminus F) \geq 2p + 2 - |F|$$

This yields,

$$x^k(\delta(V_0,\ldots,V_p)\setminus F) \geq p - \frac{|F|}{2}.$$

which means that any solution of MSTSPOC(G, K, T) does not belong to $F^k_{\Pi,F}$. Hence, $F^k_{\Pi,F}$ cannot be facet-defining.

$ii)$ Now, suppose that V_i contains 3 consecutive terminals, say t_1, t_2 and t_3 and that $S_k \nsubseteq V_i$. Consider a Steiner node $s \in V_0$ (s can also be chosen in any other V_j, $j \neq i$ such that $V_j \cap S_k \neq \emptyset$). Let $f = st_2$. Here f cannot be considered in any solution of the face $F^k_{\Pi,F}$. The proof is similar to the previous case. By the use of edge f, we have $x^k(\delta(V_i)) \geq 4$). Consequently, every solution of MSTSPOC inducing a point in $F^k_{\Pi,F}$ satisfies $x^k_f = 0$. This implies that $F^k_{\Pi,F}$ is not facet defining for MSTSPOC(G, K, T).

$iii)$ The proof of this case is similar to that of $ii)$.

2) In the sequel, one can suppose that condition 1) is satisfied. Suppose that F contains an edge $e = t_i w_j$ between two non-consecutive terminals t_i and w_j. Clearly, the following inequalities are valid for MSTSPOC(G, K, T)

$$\begin{array}{rll} x^k(\delta(V_i)) & \geq 2 & \text{for all } \ i = 1, \ldots, p, \\ -x^k(f) & \geq -1 & \text{for all } \ f \in F \setminus \{e\}, \\ x^k(g) & \geq 0 & \text{for all } \ g \in \delta(W) \setminus F, \\ -x^k(e) & = 0 & . \end{array}$$

By summing these inequalities, we obtain

$$2x^k(\delta(V_0, \ldots, V_p) \setminus F) \geq 2p - (|F| - 1).$$

By dividing by 2, we obtain

$$x^k(\delta(V_0, \ldots, V_p) \setminus F) \geq p - \lfloor \frac{|F|}{2} \rfloor.$$

As a consequence, $F^k_{\Pi,F}$ cannot be facet-defining.

3) Suppose there are two Steiner nodes s and s' of S_k and two sets V_i and V_j, $i \neq j$ such that $s \in V_i$ and $s' \in V_j$ and V_i and V_j do not contain consecutive terminals, let $f = ss'$. In what follows, we will prove that f can never be considered in a solution of $F^k_{\Pi,F}$. First, we state the following claim.

Claim If edge f is considered in some solution whose incidence vector is in $F^k_{\Pi,F}$, then $x^k(\delta(V_i)) + x^k(\delta(V_j)) \geq 6$.

Proof of the Claim: See Appendix B, Sect. B.3. □

Using the previous Claim and the results developed in the previous sections, the following inequalities are valid for MSTSPOC(G, K, T)

$$\begin{array}{rll} x^k(\delta(V_i)) + x^k(\delta(V_j)) & \geq 6, & \\ x^k(\delta(V_l)) & \geq 2 & \text{for all } \ l \in \{1, \ldots, p\} \setminus \{i, j\}, \\ -x^k(f) & \geq -1 & \text{for all } \ f \in F, \\ x^k(g) & \geq 0 & \text{for all } \ g \in \delta(W) \setminus F. \end{array}$$

The sum of these inequalities implies

$$2x^k(\delta(V_0,\ldots,V_p)\setminus F)\geq 2(p-2)+6-|F|=2p+2-|F|.$$

By dividing by 2, we obtain

$$x^k(\delta(V_0,\ldots,V_p)\setminus F)\geq p+1-\frac{|F|}{2}.$$

This implies that

$$x^k(\delta(V_0,\ldots,V_p)\setminus F)\geq p-\lfloor\frac{|F|}{2}\rfloor+\frac{1}{2},$$

and hence (27) is not satisfied with equality.

Consequently, every solution of $F^k_{\Pi,F}$ satisfies $x^k_f=0$, yielding that $F^k_{\Pi,F}$ does not define a facet for MSTSPOC(G,K,T). □

Next, we will give sufficient conditions for inequalities (27) to be facet-defining.

Theorem 9 *Inequality (27) defines a facet for MSTSPOC(G,K,T) if*

(a) every V_i, $i=1,\ldots,p$ is such that $|V_i\cap T_k|=1$,
(b) V_0 is such that $V_0\cap T_k=\emptyset$ and $|V_0\cap S_k|\geq\lceil\frac{|F|}{2}\rceil$,
(c) F is such that
  (i) $|F|=p$ if p is odd and $|F|=p-1$ if p is even,
  (ii) $|F\cap\delta(V_i)|\leq 1$ for each $i\in\{1,\ldots,p\}$,
  (iii) for each $s_j\in V_0$, if $F\cap\delta(s_j)=\{s_ju,s_jv\}$, where $u\in V_i$ and $v\in V_j$, $i\neq j$, then V_i and V_j must contain consecutive terminals.

Proof See Appendix B, Sect. B.4. □

The families of valid inequalities, previously studied, as well as the theoretical results obtained in this section, have been used within a Branch-and-Cut algorithm whose results are presented in Taktak (2024). As pointed out in Taktak (2024), these inequalities have shown to be very effective in solving the problem for a variety of random and realistic instances.

5 Further valid inequalities

In this section, we present further families of valid inequalities. These inequalities are extensions of inequalities valid for related problems.

5.1 Generalized Steiner partition inequalities

The *partition inequalities* have got a particular interest and arise as valid inequalities for several well-known problems. Introduced by Nash-Williams for the spanning tree problem in the beginning of the 60's Nash-Williams (1961), these inequalities have been later widely studied (Chopra 1989). In the beginning of the 90's, Grötschel and Monma use partitions for the connected subgraph polytope (Grötschel and Monma 1990). Later, Stoer (1992) introduces partition inequalities for both kECON (k-Edge Connected) and kNCON (k-Node Connected) subgraph problems. The author investigates necessary conditions and sufficient conditions for these inequalities to be facet-defining for the kECON and the kNCON polytopes. The partition inequalities have also been efficient in modeling the Steiner tree problem. In Chopra and Rao (1994a, b), Chopra and Rao introduce the *Steiner partition inequalities* and study their facial aspect for the Steiner tree polytope. In Barahona and Mahjoub (1995), Barahona and Mahjoub show that the partition inequalities, together with bound, cut and odd-wheel inequalities give a complete description of the 2NCON for Halin graphs. In further work, Baïou et al. (2000) propose a separation algorithm for these inequalities based on submodular functions. For more results related to partition inequalities, the reader is referred to Baïou et al. (2011).

Thus it seems to be interesting to look for an adaptation of partition inequalities to our problem. In what follows, we give a more general class of inequalities called *Generalized Steiner Partition Inequalities*.

Proposition 5 *Consider a salesman $k \in K$ and let $\Pi = (V_1, ..., V_p)$ be a partition of V such that $|V_i \cap T_k| \geq 1, i = 1, ..., p$ ($p \geq 2$). Suppose that $V_1, ..., V_r$, $r \leq p$ contain respectively $q_i \geq 2$, $i = 1, ..., r$ non-successive terminals (or sequences of terminals).*

Let $S \subseteq S_k$ be a subset of Steiner nodes of salesman k. Then

$$x^k(\delta_{G\setminus S}(V_1, V_2, ..., V_p)) \geq (p + \sum_{i=1}^{r} q_i - r) - |S| \tag{29}$$

is valid for MSTSPOC(G, K, T).

Proof The idea of the proof is to replace each time the subset V_i of the partition Π by q_i equivalent disjoint subsets. The proof is done by induction on r.

First, notice that if $|S| \geq p + \sum_{i=1}^{r} q_i - r$, inequalities (29) are redundant with respect to the trivial inequalities (5).

In the sequel, we assume that $|S| < p + \sum_{i=1}^{r} q_i - r$.

If $r = 0$, this means that all the sets V_i, $i = 1, ..., p$ contain either only one terminal or a sequence of successive terminals. In this case, inequalities (29) are equivalent to the following ones

$$x^k(\delta_{G\setminus S}(V_1, V_2, ..., V_p)) \geq p - |S| \tag{30}$$

Consider a solution (U, I) of the problem with $U = (U_1, \ldots, U_{|T_k|})$. If U_k does not use any node of S, then U_k uses at least p edges from $\delta_{G\setminus S}(V_1, \ldots, V_p)$. As any node of S can be used to route at most one section $t_i t_{i+1}$, then U_k must intersect $\delta_{G\setminus S}(V_1, \ldots, V_p)$ in at least $p - |S|$ edges, and thus inequality (30) is satisfied.

Now, suppose that (29) is valid for $r = h > 0$ and let us prove its validity for $r = h + 1$.

We know that r ($r = h+1$) subsets of the partition contain non-successive terminals (or sequences of terminals). Recall that these sets are the r first sets of the partition Π, denoted $V_1, V_2, ..., V_r$. This means that the node sets $V_{r+1}, ...V_p$ contain only successive terminals (or sequences of successive terminals). Suppose, without loss of generality, that the terminals (or sequences of terminals) of the sets V_{r-1} and V_r are pairwise non-successive. Note that this hypothesis is not restrictive since we can always consider it by a suitable numbering of the sets $V_1, V_2, ..., V_r$.

Consequently, if we combine the sets V_{r-1} and V_r in one set called W, we get exactly $q_{r-1} + q_r$ non-successive terminals (or sequences of terminals) in W.

Now, consider the new partition $\Pi' = (V'_1, ..., V'_{p'})$ where $p' = p - 1$, obtained from Π by combining the two node sets V_{r-1} and V_r. That is, Π' is defined as follows,

$$V'_j = \begin{cases} V_j & \text{if } j \in \{1, ..., r-2\}, \\ V_{r-1} \cup V_r & \text{if } j = r-1, \\ V_{j+1} & \text{if } j \in \{r, ..., p'\}. \end{cases}$$

Observe that by construction, partition Π' contains exactly $r' = r - 1$ node sets $V'_1, V'_2, ..., V'_{r'}$ such that $V'_i, i = 1, ...r'$ contains $q'_i, i = 1, ...r'$ non-successive terminals (or sequences of terminals). Remark also that $q'_i = q_i$ for all $i = 1, ...r' - 1$ and $q'_{r'} = q_{r-1} + q_r$. Since by hypothesis, (29) is assumed to be valid for the rank h and $r' = r - 1 = h + 1 - 1 = h$, we can write

$$x^k(\delta_{G\setminus S}(V'_1, V'_2, ..., V'_{p'})) \geq (p' + \sum_{i=1}^{r'} q'_i - r') - |S| \tag{31}$$

Recall that by construction, the terminals of V_{r-1} and V_r are pairwise non-successive. This means that there are no edges linking V_{r-1} to V_r in any solution, and hence $x^k(\delta_{G\setminus S}(V_{r-1}, V_r)) = 0$. We can deduce, in consequence, that:

$$x^k(\delta_{G\setminus S}(V'_1, V'_2, ..., V'_{p'})) = x^k(\delta_{G\setminus S}(V_1, V_2, ..., V_p)) \tag{32}$$

Moreover, we know that

$$\begin{aligned} (p' + \sum_{i=1}^{r'} q'_i - r') - |S| &= (p - 1 + \sum_{i=1}^{r-2} q_i + q'_{r-1} - (r-1)) - |S| \\ &= (p - 1 + \sum_{i=1}^{r-2} q_i + (q_{r-1} + q_r) - r + 1) - |S| \\ &= (p + \sum_{i=1}^{r} q_i - r) - |S|. \end{aligned}$$

Consequently,

$$(p' + \sum_{i=1}^{r'} q'_i - r') - |S| = (p + \sum_{i=1}^{r} q_i - r) - |S|. \tag{33}$$

The result follows from (32) and (33). □

Notice that the Generalized Steiner Partition inequalities written for particular values of p, r, and $|S|$, coincide with some known inequalities in the literature.

- If $p = 2$, $r = 0$ and $|S| = 1$, denote $W = V_1$ ($\overline{W} = V_2$) and $S = \{s\}$. The Generalized Steiner Partition inequalities (29) are hence equivalent to

  $$x^k(\delta_{G\backslash s}(W)) \geq 1. \tag{34}$$

  Inequalities (34) are known as the *node cut constraints* for the 2NCON problem (Stoer 1992).
- If $p > 2$ and $r = 0$, the Generalized Steiner Partition inequalities (29) are equivalent to

  $$x^k(\delta_{G\backslash S}(V_1, V_2, ..., V_p)) \geq p - |S| \tag{35}$$

  Inequalities (35) known as the *node partition constraints* for the *k*NCON problem (Stoer 1992).

In Appendix C, we investigate the relationship between inequalities (29) and inequalities (24), (3) and (5).

5.2 Generalized disjunction inequalities

In this section, we introduce further valid inequalities which, as inequalities (3) and inequalities (26), come from the disjunction constraint in the problem.

Proposition 6 *Consider a salesman $k \in K$. Let $W \subset V$ and $F \subseteq \delta(W)$ such that $|F|$ is odd. Then*

$$x^k(E(W)) + x^k(F) \leq |W| + \left\lfloor \frac{|F|}{2} \right\rfloor \tag{36}$$

is valid for MSTSPOC(G, K, T).

Proof We prove the validity of inequalities (36) for MSTSPOC(G, K, T) using a Chvátal-Gomory procedure.

Clearly, the following inequalities are valid for MSTSPOC(G, K, T)

$$\begin{array}{lll} x^k(\delta(v_i)) & \leq 2 & \text{for all } v_i \in W, \\ x^k(f) & \leq 1 & \text{for all } f \in F, \\ -x^k(g) & \leq 0 & \text{for all } g \in \delta(W) \setminus F. \end{array}$$

By summing these inequalities, we obtain

$$2(x^k(E(W)) + x^k(F)) \leq 2|W| + |F|$$

Now, by dividing by 2 and rounding up the right-hand side, we obtain

$$x^k(E(W)) + x^k(F) \leq |W| + \lfloor \frac{|F|}{2} \rfloor.$$

□

Inequalities (36) will be called *generalized disjunction inequalities*.

Notice that these inequalities look like the *blossom inequalities* for the TSP which coincide with the 2-*matching inequalities* introduced by Edmonds (1965) in the context of matching problems (Edmonds 1965).

In the following section, we will get more profit from this relationship and propose to our problem an analogue of the well-known *Comb inequalities* of the TSP.

5.3 Steiner comb inequalities

Comb inequalities have been first discovered by Chvátal (1973) in the mid-1970 s. After that, Grötschel and Padberg (1979) proposed a generalization of these inequalities.

The name "comb" comes from the form of these inequalities. Indeed, the vertices of the graph are partitioned in subsets H and T_1, ..., T_{2k+1}, where H is called the *handle* and T_1, ..., T_{2k+1} are the *teeth*.

This type of inequalities have interested many researchers who show that they are a powerful source of cutting planes for some classical problems and particularly the TSP. The efficiency of the combs was first shown (Grötschel 1980) by finding the optimal tour through 120 German cities, using a very small number of inequalities. Comb inequalities have been also shown effective for survivable network design problems. In Stoer (1992), Stoer introduces several classes of comb inequalities for 2NCON, 2ECON, and the kNCON problems. A deep facial investigation of the proposed inequalities is also held by the author.

Proposition 7 *Consider a salesman $k \in K$. Consider a family of subsets of V, H called the handle and $T_{th_1}, T_{th_2}, ..., T_{th_p}$ called the teeth such that*

1. *$p \geq 3$ and odd,*
2. *for every two disjoint teeth T_{th_i} and T_{th_j}, $T_{th_i} \cap T_{th_j} = \emptyset$,*
3. *for each tooth T_{th_j}, $|T_{th_j} \cap T_k| \geq 2$,*
4. *for each tooth T_{th_j}, $H \cap T_{th_j} \neq \emptyset$, $T_{th_j} \setminus H \neq \emptyset$, and one of the following conditions is satisfied*
   - *a) if $(H \cap T_{th_j}) \cap T_k = \emptyset$, then $T_{th_j} \setminus H$ contains at least 2 non successive terminals (or sequences of terminals),*
   - *b) if $(T_{th_j} \setminus H) \cap T_k = \emptyset$, then $H \cap T_{th_j}$ contains at least 2 non successive terminals (or sequences of terminals),*

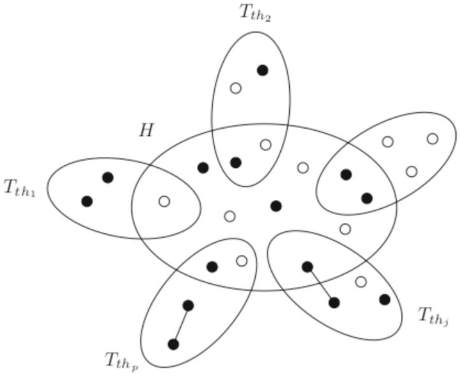

Fig. 11 Steiner Comb configuration

c) $(H \cap T_{th_j}) \cap T_k \neq \emptyset$ *and* $(T_{th_j} \backslash H) \cap T_k \neq \emptyset$.

Then

$$x^k(\delta(H)) + \sum_{i=1}^{p} x^k(\delta(T_{th_j})) \geq 3p + 1 \tag{37}$$

is valid for MSTSPOC(G, K, T).

Proof The proof of the validity here is in the same spirit as the one given for the general comb inequalities for the TSP Applegate et al. (2007).

Let us define for each $i = 1, \ldots, p$ the parameter c_i as follows:

$$c_i = \begin{cases} 1 \text{ if } x^k \text{ contains an edge between } H \cap T_{th_j} \text{ and } T_{th_j} \setminus H \\ 0 \text{ otherwise} \end{cases}$$

By condition 2) the teeth are pairwise disjoint and we have $x^k(\delta(H)) \geq \sum_{i=1}^{p} c_i$. In addition, by definition of parameter c_i, we have $\sum_{i=1}^{p} c_i \leq p$. Since $x^k(\delta(H))$ is even and p is odd, we conclude that:

$$x^k(\delta(H)) \geq 2\sum_{i=1}^{p} c_i - p + 1 \tag{38}$$

Moreover, by conditions 3) and 4), we can write for each tooth T_{th_j}

$$x^k(\delta(T_{th_j})) \geq 4 - 2ci \tag{39}$$

By (39) for all $i = 1, \ldots, p$ together with (38), we obtain

$$x^k(\delta(H)) + \sum_{i=1}^{p} x^k(\delta(T_{th_j})) \geq 2\sum_{i=1}^{p} c_i - p + 1 + \sum_{i=1}^{p} (4 - 2c_i).$$

Notice that the parameter c_i will disappear since quantity $2\sum_{i=1}^{p} c_i$ will be simplified. This yields

$$x^k(\delta(H)) + \sum_{i=1}^{p} x^k(\delta(T_{th_j})) \geq -p + 1 + \sum_{i=1}^{p} 4 = 3p + 1.$$

□

6 Conclusion

In this paper, we have studied the Multiple Steiner TSP with Order Constraints (MSTSPOC). We have considered the ILP formulation proposed for the problem in Borne et al. (2013). We have investigated the associated polytope and studied the facial aspect of its basic constraints. We have also described new families of valid inequalities, and stated necessary conditions and sufficient conditions for these inequalities to be facet-defining. Further classes of valid inequalities inspired from the TSP ones have also been presented. The theoretical results presented in this paper are computationally validated in Taktak (2024).

Several future research lines are interesting to consider. First, we would like to investigate the facial aspect of the Generalized Disjunction Inequalities and Steiner Comb Inequalities described in the last section. It would also be interesting to study some extensions of our problem. We may think about adding hop-constraints for the section paths. That is the length of the path between two consecutive terminals of a given salesman is not allowed to exceed a threshold of hops, which may have many practical interpretations.

A Appendix: Facial investigation

A.1 Proof of Theorem 3

Let F_e be the corresponding induced face, that is

$$F_e = \{(x, y) \in \text{MSTSPOC}(G, K, T) : y_e = 1\}.$$

Denote inequality $y_e \leq 1$ by $ax + by \leq \alpha$. Let $rx + qy \leq \beta$ be a valid inequality defining a facet F of MSTSPOC(G, K, T). Assume that $F_e \subseteq F$. We prove that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^j, j \in K)$, $\lambda^j \in \mathbb{R}^{|T_j|+p_j}$ for $j \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$ (where $r = (r^1, r^2, \ldots, r^{|K|})$ with $r^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ and M is the matrix of equations defined above). Note here that $b_{e'} = 0$ for all $e' \in E \setminus \{e\}$. Moreover, $a = (a^1, a^2, \ldots, a^{|K|})$ is such that $a^i \in \mathbb{R}^m$ with $a^i = 0$, $i = 1, \ldots, |K|$.

In the sequel, we will distinguish two cases.

Case 1. $e \in I^0$.

Suppose $e = t_1t_2$ is an edge between two consecutive terminals t_1 and t_2 of a salesman $k \in K$. We first show that every edge f in $E \setminus I^0$ has a coefficient q_f

equal to 0. Consider the solution (U^0, I^0) and let (U^1, I^1) be the solution defined by $U^1 = U^0$ and $I^1 = I^0 \cup \{f\}$. It is clear that (x^{U^0}, y^{I^0}) and (x^{U^1}, y^{I^1}) are in F_e and hence in F. Therefore, $rx^{U^0} + qy^{U^0} = rx^{U^0} + qy^{U^0} + q_f$ and thus $q_f = 0$. As f is arbitrary in $E \setminus I^0$, this implies that

$$q_f = 0 \qquad \text{for all } f \in E \setminus I^0. \tag{40}$$

Now, consider a salesman $l \in K$, and let $t_i t_{i+1} \in I^0 \backslash \{e\}$. Consider a Steiner node s of S_l and let (U^2, I^2) be the solution given by $U_l^2 = (U_l^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, st_{i+1}\}$, $U_j^2 = U_j^0$, $j \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^2 = \bigcup_{j \in K} U_j^2$. And let (U^3, I^3) be the solution defined by $U^3 = U^2$ and $I^3 = I^2 \cup \{t_i t_{i+1}\}$. As (x^{U^2}, y^{I^2}) and (x^{U^3}, y^{I^3}) are in F_e and thus in F, we have $rx^{U^2} + qy^{I^2} = rx^{U^2} + qy^{I^2} + q_{t_i t_{i+1}}$, which implies that $q_{t_i t_{i+1}} = 0$. As t_i and t_{i+1} are arbitrary in T_l, and l is arbitrary in K, we obtain that

$$q_f = 0 \qquad \text{for all } f \in I^0 \setminus \{e\}. \tag{41}$$

Next, we will establish some relations between the components of vector r.

Consider a salesman $l \in K$ and let s and s' be two Steiner nodes of S_l. Consider again the solution (U^0, I^0) and let (U^4, I^4) be the solution such that $U_l^4 = U_l^0 \cup \{ss'\}$, $U_j^4 = U_j^0$ for $j \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^4 = \bigcup_{j \in K} U_j^4$. Clearly (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) are both in F_e and hence in F. Consequently, $rx^{U^0} + qy^{U^0} = rx^{U^0} + qy^{U^0} + r_{ss'}^l + q_{ss'}$. By (40), it follows that $r_{ss'}^l = 0$. And therefore,

$$r_{ss'}^l = 0 \qquad \text{for all } s, s' \in S_l. \tag{42}$$

Now, consider $l \in K$, $l \neq k$ and let t_i and t_{i+1} be two consecutive terminals of T_l and s a Steiner node of S_l. Let (U^5, I^5) be given as follows $U_l^5 = (U_l^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, st_{i+1}\}$, $U_j^5 = U_j^0$ for $j \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^5 = \bigcup_{j \in K} U_j^5$. Obviously (x^{U^0}, y^{I^0}) and (x^{U^5}, y^{I^5}) are in F_e and then in F. This means that $rx^{U^0} + qy^{I^0} = rx^{U^0} + qy^{I^0} - r_{t_i t_{i+1}}^l + r_{t_i s}^l + r_{st_{i+1}}^l - q_{t_i t_{i+1}} + q_{t_i s} + q_{st_{i+1}}$. By (40), $q_{t_i s} = q_{st_{i+1}} = 0$. Moreover, as $l \neq k$, by (41), $q_{t_i t_{i+1}} = 0$. This yields $r_{t_i t_{i+1}}^l = r_{t_i s}^l + r_{st_{i+1}}^l$. As salesman l and nodes s, t_i, t_{i+1} are arbitrary, we obtain that

$$r_{t_i t_{i+1}}^l = r_{t_i s}^l + r_{st_{i+1}}^l \qquad \text{for all } \ t_i, t_{i+1} \in T_l, \ s \in S_l, l \in K \setminus \{k\}. \tag{43}$$

Similarly, we also obtain that

$$r_{t_i t_{i+1}}^k = r_{t_i s}^k + r_{st_{i+1}}^k \ \text{ for all } \ t_i, t_{i+1} \in T_k, \ s \in S_k, \ t_i t_{i+1} \neq e. \tag{44}$$

Now, we will prove that the previous relation remains valid even when $t_i t_{i+1} = e$. Consider a Steiner node s of S_k and let (U^6, I^6) be the solution defined by $U_k^6 = (U_k^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, st_{i+1}\}$, $U_j^6 = U_j^0$ for $j \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^6 = (\bigcup_{j \in K} U_j^6) \cup \{e\}$. We have that (x^{U^0}, y^{I^0}) and (x^{U^6}, y^{I^6}) are in F_e and thus in

F. This implies that $rx^{U^0} + qy^{I^0} = rx^{U^0} + qy^{I^0} - r^k_{t_i t_{i+1}} + r^k_{t_i s} + r^k_{s t_{i+1}} - q_{t_i t_{i+1}} + q_{t_i s} + q_{s t_{i+1}} + q_{t_i t_{i+1}}$ (recall that $e = t_i t_{i+1}$). By (40) it follows that $q_{t_i s} = q_{s t_{i+1}} = 0$, implying that $r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{s t_{i+1}}$. Hence,

$$r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{s t_{i+1}} \text{ for all } s \in S_k,\ t_i t_{i+1} = e. \tag{45}$$

From (44) and (45), we get

$$r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{s t_{i+1}} \text{ for all } t_i, t_{i+1} \in T_k,\ s \in S_k. \tag{46}$$

In what follows, we will prove that all the edges linking a terminal to Steiner nodes of a salesman l have the same coefficient in r^l.

First, suppose that $l \neq k$ and consider two Steiner nodes s and s' in S_l. consider again the solution (U^5, I^5) given previously and denote by (U^7, I^7) the solution given as follows $U^7_l = (U^5_l \backslash \{st_i\}) \cup \{ss', s't_i\}$, $U^7_j = U^5_j$ for each $j \in \{1, \ldots, |K|\} \backslash l$ and $I^7 = \bigcup_{j \in K} U^7_j$. It is clear that (x^{U^7}, y^{I^7}) is in F_e and thus in F. The incidence vectors of (U^5, I^5) and (U^7, I^7) satisfy that equation $rx + qy = \beta$. This, together with equations (40), allow to write $rx^{U^5} = rx^{U^7} = rx^{U^5} - r^l_{st_i} + r^l_{ss'} + r^l_{s't_i}$. Hence, $r^l_{st_i} = r^l_{ss'} + r^l_{s't_i}$. By (42), we obtain that $r^l_{st_{i+1}} = r^l_{s't_i}$

Similarly, we can show that, if $l = k$, $r^k_{st_i} = r^k_{s't_i}$.

Thus, we get

$$r^l_{st_i} = r^l_{s't_i} = \lambda^l_1(t_i) \quad \begin{array}{l} \text{for all } t_i \in T_l,\ s, s' \in S_l, l \in K, \\ \text{for some } \lambda^l_1(t_i) \in \mathbb{R}. \end{array} \tag{47}$$

Now, let $\rho = q_e$ and $\lambda^l = (\lambda^l_1, \lambda^l_2)$, $l \in K$ such that $\lambda^l_1 = (\lambda^l_1(t_i), t_i \in T_k)$ where $\lambda^l_1(t_i)$ is as given by (47) and $\lambda^l_2 = (\lambda^l_2(uv),\ u, v \in T_k, uv \notin U^0_l)$ such that $\lambda^l_2(uv) = r^l_{uv} - \lambda^l_1(u) - \lambda^l_1(v)$, $l \in K$.

The coefficients r^l_{uv} for all $uv \in E$ and $l \in K$ can then be expressed in terms of λ^l_1 and λ^l_2 as follows

$$r^l_{uv} = \begin{cases} \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_i t_{i+1},\ t_i, t_{i+1} \in T_l, \\ \lambda^l_1(u) & \text{if } u \in T_l,\ v \in S_l, \\ \lambda^l_2(uv) + \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_i t_j,\ t_i, t_j \in T_l :\ j > i, \\ 0 & \text{if } uv = s_i s_j,\ s_i, s_j \in S_l :\ j \neq i, \end{cases}$$

yielding $r^l = \lambda^l_1 M^l_1 + \lambda^l_2 M^l_2$ for all $l \in K$ as desired. Consequently, we have $q = \rho b$ and $r = \rho a + \lambda M$.

Case 2. The remaining case (i.e., $e \in E \backslash I^0$) is similarly proved.

A.2 Proof of Theorem 4

Let $F_W^{k,j}$ be the face induced by the section cut inequalities (2) corresponding to k, j and W, that is

$$F_W^{k,j} = \{(x, y) \in \mathrm{MSTSPOC}(G, K, T) : \sum_{e \in \delta_{G^{k,j}}(W)} x_e^k = 1\}.$$

Neccessity.

Assume for instance that $W \cap S_k = \emptyset$. Thus, W is reduced to a single node, namely terminal t_j. Consider a solution (U, I) of MSTSPOC with $U = (U_1, U_2, \ldots, U_{|K|})$ the edge sets corresponding to the salesmen $1, \ldots, |K|$. Let $U_{k,j}$ be the restriction of U_k on G_j^k. By constraints (2), U_k must have two edges incident to t_j. Hence, $U_{k,j}$ must have exactly one edge incident to t_j. As $W \cap S_k = \emptyset$, the cut $\delta_{G_j^k}(W)$ is reduced to that edge. And therefore, the incidence vector of (U, I), (x^U, y^I) belong to $F_W^{k,j}$. But, this implies that $F_W^{k,j} \in \mathrm{MSTSPOC}(G, K, T)$, and hence it cannot be facet defining.

The case where $\overline{W} \cap S_k = \emptyset$ is similar.

Sufficiency.

Throughout the proof, we will suppose that W and $\overline{W}$ contain each at least one Steiner node of S_k.

Denote inequality (2) corresponding to k, j and W by $ax + by \le \alpha$ and let $F_W^{k,j} = \{(x, y) \in \mathrm{MSTSPOC}(G, K, T) : ax + by = \alpha\}$. Let $rx + qy \le \beta$ be a valid inequality defining a facet F of $\mathrm{MSTSPOC}(G, K, T)$ such that $F_W^{k,j} \subseteq F$. In the following, we prove that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^l, l \in K)$, $\lambda^l \in \mathbb{R}^{|T_l|+p_l}$ for $l \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$ (where $r = (r^1, r^2, \ldots, r^{|K|})$ with $r^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ and M is the matrix of equations defined above). Note here that $a = (a^1, a^2, \ldots, a^{|K|})$ is such that $a^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ with $a^i = 0$ for $i \in \{1, \ldots, |K|\} \backslash \{k\}$, $a_e^k \neq 0$ for every $e \in \delta_{G_j^k}(W)$ and $a_{e'}^k = 0$ for every $e' \in E \backslash \delta_{G_j^k}(W)$. Also, note that $b = 0$.

First, we prove that $q = 0$.

Consider the solution (U^0, I^0) and let $e \in E \backslash I^0$ be an arbitrary edge. Let (U^1, I^1) be the solution given by $I^1 = I^0 \cup \{e\}$ and $U^1 = U^0$. Both (U^0, I^0) and (U^1, I^1) induce solutions in $F_W^{k,j}$. This implies that their incidence vectors satisfy equation $rx + qy = \beta$. Consequently, $rx^{U^0} + qy^{I^0} = qx^{U^0} + qy^{I^0} + q_e$, implying that $q_e = 0$. Since e is arbitrarily chosen in $E \setminus I^0$, we have

$$q_e = 0 \qquad \text{for all } e \in E \setminus I^0. \tag{48}$$

Similarly, we can show that

$$q_e = 0 \qquad \text{for all } e \in I^0. \tag{49}$$

By (48) and (49) we then have

$$q_e = 0 \qquad \text{for all } e \in E. \tag{50}$$

Next, we will establish some relations between the components of vector r.

First, we consider the edges between Steiner nodes.

Let $l \in K\backslash\{k\}$ and s and s' be two Steiner nodes of S_l. Consider the solution (U^0, I^0) and let (U^4, I^4) be the solution defined by $U_l^4 = U_l^0 \cup \{ss'\}$, $U_p^4 = U_p^0$ for all $p \in \{1, \ldots, |K|\}\backslash l$ and $I^4 = \bigcup_{p\in K} U_p^4$. Clearly, (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) are both in $F_W^{k,j}$ and thus in F. This implies that (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) satisfy $rx+qy = \beta$. Since $q = 0$, we have $r^k x^{U^0} = r^k x^{U^0} + r^l_{ss'}$, which yields $r^l_{ss'} = 0$. As salesman l and nodes s and s' are all arbitrary, we have

$$r^l_{ss'} = 0 \qquad \text{for all } \ s, s' \in S_l, \ l \in K \setminus \{k\}. \tag{51}$$

Similarly, we obtain that

$$r^k_{ss'} = 0 \qquad \text{for all } \ s, s' \in S_k, \ ss' \notin \delta_{G_j^k}(W). \tag{52}$$

Now, we will consider the edges between terminals.

Consider a salesman $l \in K \setminus \{k\}$ and let t_i and t_{i+1} be two terminals of T_l. Let s be a Steiner node of S_l. Consider the solution (U^6, I^6) given by $U_l^6 = (U_l^0\backslash\{t_i t_{i+1}\}) \cup \{t_i s, st_{i+1}\}, U_p^6 = U_p^0$, for each $p \in \{1, \ldots, |K|\}\backslash l$ and $I^6 = \bigcup_{p\in K} U_p^6$. As (x^{U^6}, y^{I^6}) is in $F_W^{k,j}$ and hence in F, it follows that $rx^{U^0} = rx^{U^0} - r^l_{t_i t_{i+1}} + r^l_{t_i s} + r^l_{st_{i+1}}$, implying that $r^l_{t_i t_{i+1}} = r^l_{t_i s} + r^l_{st_{i+1}}$. Salesman l as well as nodes s, t_i and t_{i+1} are all arbitrary, therefore,

$$r^l_{t_i t_{i+1}} = r^l_{t_i s} + r^l_{st_{i+1}} \qquad \text{for all } t_i, t_{i+1} \in T_l, \ s \in S_l, \ l \in K \setminus \{k\}. \tag{53}$$

If t_i and t_{i+1} are terminals of T_k such that $t_i t_{i+1} \neq t_j t_{j+1}$, we can also prove along the same line that

$$r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{st_{i+1}} \qquad \text{for all } t_i, t_{i+1} \in T_k, \ s \in S_k, \ t_i t_{i+1} \neq t_j t_{j+1}. \tag{54}$$

Consider now two Steiner nodes s and s' of S_k such that $s \in W$ and $s' \in \overline{W}$ (recall that $W \cap S_k \neq \emptyset$ and $\overline{W} \cap S_k \neq \emptyset$). Let (U^7, I^7) be the solution defined as follows: $U_k^7 = (U_k^0\backslash\{t_j t_{j+1}\}) \cup \{t_j s, ss', s't_{j+1}\}$, $U_p^7 = U_p^0$, for each $p \in \{1, \ldots, |K|\}\backslash k$ and $I^7 = \bigcup_{p\in K} U_p^7$. As (x^{U^0}, y^{I^0}) and (x^{U^7}, y^{I^7}) are both in $F_W^{k,j}$ and thus in F, we have $rx^{U^0} = rx^{U^7} = rx^{U^0} - r^k_{t_j t_{j+1}} + r^k_{t_j s} + r^k_{ss'} + r^k_{s't_{j+1}}$, yielding $r^k_{t_j t_{j+1}} = r^k_{t_j s} + r^k_{ss'} + r^k_{s't_{j+1}}$. Hence, we have

$$r^k_{t_j t_{j+1}} = r^k_{t_j s} + r^k_{ss'} + r^k_{s't_{j+1}} \qquad \text{for all } s, s' \in S_k, \ s \in W \text{ and } s' \in \overline{W}. \tag{55}$$

In what follows, we will look at the coefficients of edges between terminals and Steiner nodes.

To this end, consider first a salesman $l \in K\backslash\{k\}$ and let (U^8, I^8) be the solution obtained from (U^6, I^6) as follows, $U_l^8 = (U_l^6\backslash\{st_{i+1}\}) \cup \{ss', s't_{i+1}\}$, $U_p^8 = U_p^6$ for each $p \in \{1, \ldots, |K|\} \backslash l$ and $I^8 = \bigcup_{j \in K} U^8_j$, where s and s' are Steiner nodes of S_l. Since (x^{U^6}, y^{I^6}) and (x^{U^8}, y^{I^8}) are both in $F^{k,j}_W$ and thus in F, this implies that $rx^{U^6} = rx^{U^6} - r^l_{st_{i+1}} + r^l_{ss'} + r^l_{s't^k_{i+1}}$. By (51) it follows that $r^l_{st_{i+1}} = r^l_{s't_{i+1}}$. As salesman l and nodes s, s' and t_i are all arbitrary, we have

$$r^l_{st_i} = r^l_{s't_i} = \lambda^l_1(t_i) \quad \begin{array}{l} \text{for all } \ t_i \in T_l, \ s, s' \in S_l, l \in K \setminus \{k\}, \\ \text{for some } \lambda^l_1(t_i) \in \mathbb{R}. \end{array} \tag{56}$$

If $l = k$, along the same way we obtain that

$$r^k_{st_i} = r^k_{s't_i} = \lambda^k_1(t_i) \quad \begin{array}{l} \text{for all } \ t_i \in T_k, \ s, s' \in S_k, ss' \notin \delta_{G^k_j}(W) \\ \text{for some } \lambda^k_1(t_i) \in \mathbb{R}. \end{array} \tag{57}$$

Suppose now that s and s' are Steiner nodes of S_k such that $ss' \notin \delta_{G^{k,j}}(W)$. Consider the solution (U^9, I^9) defined as follows, $U^9_k = (U^0_k \backslash \{t_j t_{j+1}\}) \cup \{t_j s, st_{j+1}\}$, $U^9_p = U^0_p$, for each $p \in \{1, ..., K\} \backslash k$ and $I^9 = \bigcup_{p \in K} U^9_p$. We also define (U^{10}, I^{10}) by: $U^{10}_k = (U^9_k \backslash \{t_j s\}) \cup \{t_j s', ss'\}$, $U^{10}_p = U^9_p$, for each $p \in \{1, ..., K\} \backslash k$ and $I^{10} = \bigcup_{p \in K} U^{10}_p$. Solutions (x^{U^9}, y^{I^9}) and $(x^{U^{10}}, y^{I^{10}})$ are both in $F^{k,j}_W$ and thus in F. As a consequence, we have $rx^{U^9} = rx^{U^{10}} = rx^{U^{10}} - r^k_{t_j s} + r^k_{t_j s'} + r^k_{ss'}$, yielding $r^k_{t_j s} = r^k_{t_j s'} + r^k_{ss'}$. By (52), it follows that $r^k_{t_j s} = r^k_{t_j s'}$. Similarly, we can show that $r^k_{t_{j+1} s} = r^k_{t_{j+1} s'}$. And since s and s' are arbitrary in S_k, we have

$$r^k_{t_i s} = r^k_{t_i s'} = \lambda^k_1(t_i) \quad \begin{array}{l} \text{for all } s, s' \in S_k, \ ss' \notin \delta_{G^{k,j}}(W), \ t_i \in \{t_j, t_{j+1}\}, \\ \text{for some } \lambda^k_1(t_i) \in \mathbb{R}. \end{array} \tag{58}$$

Now, suppose that $ss' \in \delta_{G^k_j}(W)$ with $s' \in W$ and $s \in \overline{W}$. Along the same line, we can prove that $r^k_{t_j s} = r^k_{t_j s'} + r^k_{ss'}$ and $r^k_{t_{j+1} s} = r^k_{t_{j+1} s'} + r^k_{ss'}$. As s and s' are arbitrary, we have the following

$$r^k_{t_i s} = r^k_{t_i s'} + r^k_{ss'} \quad \begin{array}{l} \text{for all } s, s' \in S_k, \ s' \in W, \ s \in \overline{W}, \\ t_i \in \{t_j, t_{j+1}\}. \end{array} \tag{59}$$

Now, we will go back to the case that we left in the beginning of the proof, concerning edges between Steiner nodes of S_k that belong to $\delta_{G^k_j}(W)$.

Consider salesman k and let s_1, s_2, s_3 and s_4 be Steiner nodes of S_k such that, s_1 and s_3 are in W, and s_2 and s_4 are in $\overline{W}$ (s_1 and s_3, s_2 and s_4 may be the same). By (59) and as $s_1 s_2, s_3 s_4 \in \delta_{G^k_j}(W)$, we have that, $r^k_{t_{j+1} s_1} = r^k_{t_{j+1} s_2} + r^k_{s_1 s_2}$ and $r^k_{t_{j+1} s_3} = r^k_{t_{j+1} s_4} + r^k_{s_3 s_4}$. Moreover, by (58) and since $s_1 s_3, s_2 s_4 \notin \delta_{G^k_j}(W)$, it follows that $r^k_{t_{j+1} s_1} = r^k_{t_{j+1} s_3}$ and $r^k_{t_{j+1} s_2} = r^k_{t_{j+1} s_4}$, yielding $r^k_{s_1 s_2} = r^k_{s_3 s_4}$. As s_1, s_2, s_3 and s_4 are all arbitrary in S_k, we then have

$$r^k_{s_1s_2} = r^k_{s_3s_4} = \rho \quad \begin{array}{l} \text{for all } s_1, s_2, s_3, s_4 \in S_k, \\ s_1s_2, s_3s_4 \in \delta_{G^k_j}(W) \\ \text{for some } \rho \in \mathbb{R}. \end{array} \tag{60}$$

Now, let $\rho \in \mathbb{R}$ be as given by (60) and $\lambda^l = (\lambda^l_1, \lambda^l_2)$, $l \in K$ such that $\lambda^l_1 = (\lambda^l_1(t_i), t_i \in T_k)$ where $\lambda^l_1(t_i)$ is as given by (56), (57), (58) and (59). $\lambda^l_2 = (\lambda^l_2(uv),\ u, v \in T_k, uv \notin U^0_l)$ such that $\lambda^l_2(uv) = r^l_{uv} - \lambda^l_1(u) - \lambda^l_1(v)$, $l \in K$.

Overall, the coefficients r^l_{uv} for all $uv \in E$ and $l \in K$ can then be expressed in terms of ρ, λ^l_1 and λ^l_2 as follows

$$r^l_{uv} = \begin{cases} \lambda^l_1(u) + \lambda^l_1(v) & \text{if } l \neq k,\ uv = t_it_{i+1},\ t_i, t_{i+1} \in T_l, \\ \lambda^k_1(u) + \lambda^k_1(v) & \text{if } l = k,\ uv = t_it_{i+1},\ t_i, t_{i+1} \in T_k, uv \neq t_jt_{j+1}, \\ \rho + \lambda^k_1(u) + \lambda^k_1(v) & \text{if } l = k,\ uv = t_jt_{j+1}, \\ \lambda^l_1(u) & \text{if } l \neq k,\ u \in T_l,\ v \in S_l, \\ \lambda^k_1(u) & \text{if } l = k,\ u \in T_k \setminus \{t_j, t_{j+1}\},\ v \in S_k, \\ \rho + \lambda^k_1(u) & \text{if } l = k,\ u \in T_k \setminus \{t_j, t_{j+1}\},\ v \in S_k, uv \in \delta_{G^k_j}(W), \\ \lambda^l_2(uv) + \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_it_{i'},\ t_i, t_{i'} \in T_l,\ i' > i, \\ 0 & \text{if } l \neq k \text{ and } uv = s_is_{i'},\ s_i, s_{i'} \in S_l,\ i' \neq i, \\ 0 & \text{if } l = k \text{ and } uv = s_is_{i'},\ s_i, s_{i'} \in S_k,\ s_is_{i'} \notin \delta_{G^k_j}(W), \\ \rho & \text{if } l = k \text{ and } uv = s_is_{i'},\ s_i, s_{i'} \in S_k,\ s_is_{i'} \in \delta_{G^k_j}(W). \end{cases}$$

Clearly, $r^l = \rho a^l + \lambda^l_1 M^l_1 + \lambda^l_2 M^l_2,$ for all $l \in K$. As a consequence, $r = \rho a + \lambda M$, and the proof is complete.

B Appendix: Valid Inequalities

B.1 Proof of Theorem 6 (Sufficiency)

In the sequel, we suppose that conditions 1), 2) and 3) are satisfied.

Denote inequality (24) corresponding to k and W by $ax + by \leq \alpha$. Let $rx + qy \leq \beta$ be a valid inequality defining a facet F of MSTSPOC(G, K, T). In the following, we prove that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^l, l \in K)$, $\lambda^l \in \mathbb{R}^{|T_l|+p_l}$ for $l \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$.

First, we prove that $q = 0$.

Consider the solution (U^0, I^0) and let $e \in E \setminus I^0$ be an arbitrary edge. Consider also the solution (U^1, I^1) such that $I^1 = I^0 \cup \{e\}$ and $U^1 = U^0$. Both induce vectors that are in F^k_W and thus in F. This implies that $q_e = 0$. Since e is arbitrarily chosen in $E \setminus I^0$, we then have

$$q_e = 0 \qquad \text{for all } e \in E \setminus I^0. \tag{61}$$

Now, consider the solution (U^2, I^2) obtained as follows. Consider a salesman $l \in K$ and let $e = t_it_{i+1}$ be an edge between two consecutive terminals t_i and t_{i+1} of T_l.

Let $U_l^2 = (U_l^0 \backslash \{e\}) \cup \{t_i s, st_{i+1}\}$ where $s \in S_l$ is a Steiner node of salesman l. In addition, let $U_j^2 = U_j^0$, $j = 1, \ldots, K$, $j \neq l$, and $I^2 = \bigcup_{j \in K} U_j^2$. Now, let us define the solution (U^3, I^3) given by $U^3 = U^2$ and $I^3 = I^2 \cup \{e\}$. Both are feasible solutions of MSTSPOC inducing vectors that are in F_W^k and hence satisfying equation $rx + qy = \beta$. As a consequence, we obtain that $q_e = 0$. As e is arbitrary in I^0, this yields

$$q_e = 0 \qquad \text{for all } e \in I^0. \tag{62}$$

By (61) and (62) we then have

$$q_e = 0 \qquad \text{for all } e \in E. \tag{63}$$

Next, we will examine vector r.

Consider a salesman $l \in K \setminus \{k\}$ and let s and s' be two Steiner nodes of S_l. Consider the solution (U^0, I^0) defined above and let (U^4, I^4) be the pair defined by $U_l^4 = U_l^0 \cup \{ss'\}$, $U_p^4 = U_p^0$ for all $p \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^4 = \bigcup_{p \in K} U_p^4$. It is clear that (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) are both in F_W^k and thus in F. This implies that (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) satisfy $rx + qy = \beta$. Since $q = 0$, we have $rx^{U^0} = rx^{U^4} = r^k x^{U^0} + r_{ss'}^l$, which implies $r_{ss'}^l = 0$. As salesman l and Steiner nodes s and s' are arbitrary, we have

$$r_{ss'}^l = 0 \qquad \text{for all } \ s, s' \in S_l, \ l \in K \setminus \{k\}. \tag{64}$$

Similarly, we can show that,

$$r_{ss'}^k = 0 \qquad \text{for all } \ s, s' \in S_k, \ ss' \notin \delta(W). \tag{65}$$

The case where $ss' \in \delta(W)$ for $s, s' \in S_k$ will be treated at the end of the proof.

Consider a salesman $l \in K \setminus \{k\}$ and let t_i and t_{i+1} be two terminals of T_l. Let s be a Steiner node of S_l. Consider the solution (U^6, I^6) given by $U_l^6 = (U_l^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, st_{i+1}\}$, $U_p^6 = U_p^0$, for each $p \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^6 = \bigcup_{p \in K} U_p^6$. As (x^{U^6}, y^{I^6}) is in F_W^k and hence in F, it follows that $rx^{U^0} = rx^{U^0} - r_{t_i t_{i+1}}^l + r_{t_i s}^l + r_{st_{i+1}}^l$, implying that $r_{t_i t_{i+1}}^l = r_{t_i s}^l + r_{st_{i+1}}^l$. Therefore

$$r_{t_i t_{i+1}}^l = r_{t_i s}^l + r_{st_{i+1}}^l \qquad \text{for all } t_i, t_{i+1} \in T_l, \ s \in S_l, \ l \in K \setminus \{k\}. \tag{66}$$

Now, suppose that t_i and t_{i+1} are terminals of T_k such that $t_i t_{i+1} \notin \delta(W)$. Suppose, without loss of generality, that t_i and t_{i+1} are in W. Consider a Steiner node $s \in S_k \cap W$. Along the same line, we can prove that $r_{t_i t_{i+1}}^k = r_{t_i s}^k + r_{st_{i+1}}^k$. Hence,

$$r_{t_i t_{i+1}}^k = r_{t_i s}^k + r_{st_{i+1}}^k \qquad \begin{array}{l} \text{for all } t_i, t_{i+1} \in T_k \cap W \text{ (resp. } T_k \cap \overline{W}), \\ s \in S_k \cap W \text{ (resp. } S_k \cap \overline{W}). \end{array} \tag{67}$$

Suppose now that $t_i t_{i+1} \in \delta(W)$ such that $t_i \in W$ and $t_{i+1} \in \overline{W}$. Consider two Steiner nodes s and s' of S_k such that $s \in W$ and $s' \in \overline{W}$. Let (U^7, I^7) be the solution given by $U_k^7 = (U_k^0 \backslash \{t_i t_{i+1}\}) \cup \{t_i s, ss', s' t_{i+1}\}$, $U_p^7 = U_p^0$, for each $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^7 = \bigcup_{p \in K} U_p^7$. As (x^{U^0}, y^{I^0}) and (x^{U^7}, y^{I^7}) are both in F_W^k and thus in F, we have $rx^{U^0} = rx^{U^7} = rx^{U^0} - r^k_{t_i t_{i+1}} + r^k_{t_i s} + r^k_{ss'} + r^k_{s' t_{i+1}}$, yielding $r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{ss'} + r^k_{s' t_{i+1}}$. As nodes t_i, t_{i+1}, s and s' are all arbitrary, we then have

$$r^k_{t_i t_{i+1}} = r^k_{t_i s} + r^k_{ss'} + r^k_{s t_{i+1}} \qquad \begin{array}{l} \text{for all } t_i, t_{i+1} \in T_k,\ t_i \in W,\ t_{i+1} \in \overline{W}, \\ \text{for all } s, s' \in S_k,\ s \in W,\ s' \in \overline{W}. \end{array} \tag{68}$$

In the following, we will look at the coefficients of edges between a terminal and Steiner nodes.

To this end, consider a salesman $l \in K \backslash \{k\}$ and let (U^8, I^8) be the pair obtained from (U^7, I^7) as follows: $U_l^8 = (U_l^7 \backslash \{s t_{i+1}\}) \cup \{ss', s' t_{i+1}\}$, $U_p^8 = U_p^7$ for each $p \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^8 = \bigcup_{j \in K} U_j^8$, where s and s' are Steiner nodes of S_l. Since (x^{U^7}, y^{I^7}) and (x^{U^8}, y^{I^8}) are both in $F_W^{k,j}$ and thus in F, this implies $rx^{U^7} = rx^{U^8} = rx^{U^7} - r^l_{s t_{i+1}} + r^l_{ss'} + r^l_{s' t^k_{i+1}}$. By (64), it follows that $r^l_{s t_{i+1}} = r^l_{s' t_{i+1}}$. As salesman l, and nodes t_i, s and s' are arbitrary, we have

$$r^l_{s t_i} = r^l_{s' t_i} = \lambda_1^l(t_i) \quad \begin{array}{l} \text{for all } t_i \in T_l,\ s, s' \in S_l,\ l \in K \setminus \{k\}, \\ \text{for some } \lambda_1^l(t_i) \in \mathbb{R}. \end{array} \tag{69}$$

Similarly, we can show that

$$r^k_{t_i s} = r^k_{t_i s'} = \lambda_1^l(t_i) \quad \begin{array}{l} \text{for all } t_i, t_{i+1} \in T_k \cap W \text{ (resp. } T_k \cap \overline{W}), \\ \text{for all } s, s' \in S_k \cap W \text{ (resp. } S_k \cap \overline{W}), \\ \text{for some } \lambda_1^l(t_i) \in \mathbb{R}. \end{array} \tag{70}$$

Now, consider again the terminal t_i and suppose now that $\{t_{i-1} t_i, t_i t_{i+1}\} \nsubseteq \delta(W)$. Consider the two Steiner nodes s and s' and suppose that t_i and the set $\{s, s'\}$ are situated on different sides of the cut $\delta(W)$. That is $t_i s \in \delta(W)$ and $s t_{i+1} \in \delta(W)$. The same reasoning as the previous ones enables us to write

$$r^k_{s t_i} = r^k_{s' t_i} = \lambda_1^k(t_i) \quad \begin{array}{l} \text{for all } t_i \in T_k,\ s, s' \in S_k,\ s t_i \in \delta(W) \text{ and } s' t_i \in \delta(W), \\ \text{for some } \lambda_1^l(t_i) \in \mathbb{R}. \end{array} \tag{71}$$

Now, we will consider the case where t_i and s' are in W and $s \in \overline{W}$. Similarly, we can have the following relation $r^k_{t_i s} = r^k_{t_i s'} + r^k_{ss'}$. And since t_i, s and s' are arbitrary, we have

$$r^k_{s t_i} = r^k_{s' t_i} + r^k_{ss'} \qquad \begin{array}{l} \text{for all } t_i \in T_k,\ s, s' \in S_k, \\ s t_i \in \delta(W) \text{ and } s' t_i \notin \delta(W). \end{array} \tag{72}$$

Now, we will go back to the case that we left in the beginning of the proof, concerning edges between Steiner nodes of S_k that belong to $\delta(W)$.

Consider salesman k, a terminal t_1 of T_k and let s_1, s_2, s_3 and s_4 be Steiner nodes of S_k such that, s_1 and s_3 are in W, and t_1, s_2 and s_4 are in $\overline{W}$ (s_1 and s_3, resp. s_2 and s_4,may be the same). By (72) and as $s_1s_2 \in \delta(W)$ and $s_3s_4 \in \delta(W)$, we have the following, $r^k_{t_1s_1} = r^k_{t_1s_2} + r^k_{s_1s_2}$ and $r^k_{t_1s_3} = r^k_{t_1s_4} + r^k_{s_3s_4}$. Moreover, by (71) and since $s_1s_3 \notin \delta(W)$ and $s_2s_4 \notin \delta(W)$, it follows that $r^k_{t_1s_1} = r^k_{t_1s_3}$ and $r^k_{t_1s_2} = r^k_{t_1s_4}$, yielding $r^k_{s_1s_2} = r^k_{s_3s_4}$. As s_1, s_2, s_3 and s_4 are all arbitrary in S_k, we then have

$$r^k_{s_1s_2} = r^k_{s_3s_4} = \rho \quad \begin{array}{l} \text{for all } s_1, s_2, s_3, s_4 \in S_k, \\ s_1, s_3 \in W \text{ and } s_2, s_4 \in \overline{W}, \\ \text{for some } \rho \in \mathbb{R}. \end{array} \tag{73}$$

Now, let $\rho \in \mathbb{R}$ be as given by (73) and $\lambda^l = (\lambda^l_1, \lambda^l_2)$, $l \in K$ such that $\lambda^l_1 = (\lambda^l_1(t_i), t_i \in T_k)$ where $\lambda^l_1(t_i)$ is as given by (69), (70) and (71). $\lambda^l_2 = (\lambda^l_2(uv),\ u, v \in T_k, uv \notin U^0_l)$ such that $\lambda^l_2(uv) = r^l_{uv} - \lambda^l_1(u) - \lambda^l_1(v)$, $l \in K$.

Overall, the coefficients r^l_{uv} for all $uv \in E$ and $l \in K$ can then be expressed in terms of ρ, λ^l_1 and λ^l_2 as follows

$$r^l_{uv} = \begin{cases} \lambda^l_1(u) + \lambda^l_1(v) & \text{if } l \neq k,\ uv = t_it_{i+1},\ t_i, t_{i+1} \in T_l, \\ \lambda^k_1(u) + \lambda^k_1(v) & \text{if } l = k,\ uv = t_it_{i+1},\ t_i, t_{i+1} \in T_k, uv \notin \delta(W), \\ \rho + \lambda^k_1(u) + \lambda^k_1(v) & \text{if } l = k,\ uv \in \delta(W), \\ \lambda^l_1(u) & \text{if } l \neq k,\ u \in T_l,\ v \in S_l, \\ \lambda^k_1(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \notin \delta(W) \\ \rho + \lambda^k_1(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \in \delta(W) \\ \lambda^l_2(uv) + \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_it_j,\ t_i, t_j \in T_l,\ j > i, \\ 0 & \text{if } l \neq k \text{ and } uv = s_is_j,\ s_i, s_j \in S_l,\ j \neq i, \\ 0 & \text{if } l = k \text{ and } uv = s_is_j,\ s_i, s_j \in S_k,\ uv \notin \delta(W), \\ \rho & \text{if } l = k \text{ and } uv = s_is_j,\ s_i, s_j \in S_k,\ uv \in \delta(W). \end{cases}$$

Clearly, $r^l = \rho a^l + \lambda^l_1 M^l_1 + \lambda^l_2 M^l_2$, for all $l \in K$. As a consequence, $r = \rho a + \lambda M$ and the result follows.

B.2 Proof of Theorem 7

Necessity

Assume for instance that V_1 contains terminals predecessor to w_{j-1} (the case when V_3 contains terminals successor to w_{j+1} can be treated in a similar way). Consider the partition $\Pi' = (V'_0, V'_1, \ldots, V'_p)$, where

$$\begin{aligned} V'_1 &= V_1 \setminus \{w_{j-l}, \ldots, w_{j-2}\}, \\ V'_4 &= V_4 \cup \{w_{j-l}, \ldots, w_{j-2}\}, \\ V'_i &= V_i, \text{ otherwise.} \end{aligned}$$

It is clear that the left-hand side of inequality (26) with respect to partition Π' is less or equal than that of partition Π. Moreover, the right-hand side of inequality (26) is the same for both partitions Π and Π'. This implies that inequality (26) induced by partition Π' dominates the one induced by partition Π. As a consequence, (26) cannot be facet defining for MSTSPOC(G, K, T).

Sufficiency

In what follows, we will assume that $V_1 \cap T_k = \{w_{j-1}\}$ and $V_3 \cap T_k = \{w_{j+1}\}$. Note here that V_1 and V_3 may contain Steiner nodes of S_k.

Denote inequality (26) by $ax + by \leq \alpha$ and let

$$F^k_{j,S} = \{(x, y) \in \text{MSTSPOC}(G, K, T) : \sum_{e \in F_{j-1} \cup F_{j+1}} x^k_e = \sum_{e \in F_j} x^k_e\}.$$

Let $rx + qy \leq \beta$ be a valid inequality defining a facet F of MSTSPOC(G, K, T) such that $F^k_{j,S} \subseteq F$. In the following, we will prove that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^l, l \in K)$, $\lambda^l \in \mathbb{R}^{|T_l|+p_l}$ for $l \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$. Notice here that $a = (a^1, a^2, \ldots, a^{|K|})$ such that $a^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ with $a^i = 0$ for $i \in \{1, \ldots, |K|\} \backslash \{k\}$, $a^k_e \neq 0$ for every $e \in F_{j-1} \cup F_j \cup F_{j+1}$ and $a^k_{e'} = 0$ for every $e' \in E \backslash (F_{j-1} \cup F_j \cup F_{j+1})$. Remark also that $b = 0$.

First, we prove that $q = 0$.

Consider the solution (U^0, I^0) and let $e \in E \setminus I^0$ be an arbitrary edge. Consider the solution (U^1, I^1) given by $U^1 = U^0$ and $I^1 = I^0 \cup \{e\}$. Note that the incidence vectors of solutions (U^0, I^0) and (U^1, I^1) are in $F^k_{j,S}$. This means that (x^{U^0}, y^{I^0}) and (x^{U^1}, y^{I^1}) satisfy equation $rx + qy = \beta$. Consequently, $rx^{U^0} + qy^{I^0} = rx^{U^1} + qy^{I^1} = qx^{U^0} + qy^{I^0} + q_e$, implying that $q_e = 0$. Since e is arbitrarily chosen in $E \backslash I^0$, we have

$$q_e = 0 \qquad \text{for all } e \in E \setminus I^0. \tag{74}$$

Now, consider the solution (U^2, I^2) obtained as follows. Consider a salesman $l \in K$ and let $e = w_i w_{i+1}$ be an edge between two consecutive terminals w_i and w_{i+1} of T_l. Let $U^2_l = (U^0_l \backslash \{e\}) \cup \{w_i s, s w_{i+1}\}$ where $s \in S_l$ is a Steiner node of salesman l (if $l = k$ s could be either in S or in $S_k \backslash S$). In addition, let $U^2_j = U^0_j$, $j = 1, \ldots, |K|$, $j \neq l$, and $I^2 = \bigcup_{j \in K} U^2_j$. Now, let us define the solution (U^3, I^3) given by $U^3 = U^2$ and $I^3 = I^2 \cup \{e\}$. The incidence vectors of both solutions (U^2, I^2) and (U^3, I^3) are in $F^k_{j,S}$, and hence they satisfy equation $rx + qy = \beta$. As a consequence, we have $rx^{U^2} + qy^{I^2} = rx^{U^3} + qy^{I^3} = rx^{U^2} + qy^{I^2} + q_e$, which implies that $q_e = 0$. Recall that I^0 is the set of edges between the consecutive terminals of all the salesmen. As e is arbitrary in I^0, this yields

$$q_e = 0 \qquad \text{for all } e \in I^0. \tag{75}$$

By (74) and (75) we then have

$$q_e = 0 \qquad \text{for all } e \in E. \tag{76}$$

Now, we will examine the coefficients between Steiner nodes.

Consider a salesman $l \in K \setminus \{k\}$ and let s and s' be two Steiner nodes of S_l. Consider the solution (U^0, I^0) defined above and let (U^4, I^4) be the pair defined by $U_l^4 = U_l^0 \cup \{ss'\}$, $U_p^4 = U_p^0$ for all $p \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^4 = \bigcup_{p \in K} U_p^4$. Clearly (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) are both in $F_{j,S}^k$ and thus in F. This implies that (x^{U^0}, y^{I^0}) and (x^{U^4}, y^{I^4}) satisfy equation $rx + qy = \beta$. Since $q = 0$, we have $rx^{U^0} = r^k x^{U^0} + r_{ss'}^l$, which yields $r_{ss'}^l = 0$. As salesman l and Steiner nodes s and s' are arbitrary, we have

$$r_{ss'}^l = 0 \qquad \text{for all } \ s, s' \in S_l, \ l \in K \setminus \{k\}. \tag{77}$$

Now, consider salesman k and suppose that s and s' are Steiner nodes of S_k such that $ss' \notin \delta(S)$. That is, ss' is an edge between two Steiner nodes which are both either in S or in $S_k \setminus S$. Let (U^5, I^5) be the solution defined as follows $U_k^5 = U_k^0 \cup \{ss'\}$, $U_p^5 = U_p^0$ for all $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^5 = \bigcup_{p \in K} U_p^5$. Obviously, (x^{U^5}, y^{I^5}) is in $F_{j,S}^k$ and hence in F. As a consequence, the incidence vectors of (U^0, I^0) and (U^5, I^5) satisfy equation $rx + qy = \beta$. Since $q = 0$, it follows that $r_{ss'}^k = 0$. As s and s' are arbitrary in S_k, we have that

$$r_{ss'}^k = 0 \qquad \text{for all } \ s, s' \in S_k, \ ss' \notin \delta(S). \tag{78}$$

Now, consider four Steiner nodes of salesman k denoted s_1, s_2, s_3 and s_4 such that $s_1, s_3 \in V_0$ and $s_2, s_4 \in V_1$ (resp. $s_2, s_4 \in V_3$). Note that s_1 may coincide with s_3, and similarly s_2 may coincide with s_4. Let (U^6, I^6) be the solution defined as follows $U_k^6 = U_k^0 \backslash w_j w_{j+1} \cup \{w_j s_1, s_1 s_2, s_2 w_{j+1}\}$, $U_p^6 = U_p^0$ for all $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^6 = \bigcup_{p \in K} U_p^6$. Consider also the solution (U^7, I^7) given by $U_k^7 = U_k^0 \backslash w_j w_{j+1} \cup \{w_j s_3, s_3 s_4, s_4 w_{j+1}\}$, $U_p^7 = U_p^0$ for all $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^7 = \bigcup_{p \in K} U_p^7$. It is clear that (x^{U^6}, y^{I^6}) and (x^{U^7}, y^{I^7}) are in $F_{j,S}^k$ and hence in F. Since $q = 0$, this implies that $rx^{U^6} = rx^{U^6} - r_{s_1 s_2}^k + r_{s_1 s_3}^k + r_{s_3 s_4}^k + r_{s_4 s_2}^k$. As $s_1 s_3 \notin \delta(S)$ and $s_4 s_2 \notin \delta(S)$, by (78) we obtain that $r_{s_1 s_2}^k = r_{s_3 s_4}^k$. As Steiner nodes s_1, s_2, s_3 and s_4 are arbitrarily chosen in S_k, we have

$$r_{s_1 s_2}^k = r_{s_3 s_4}^k = \rho \quad \begin{array}{l} \text{for all } s_1, s_2, s_3, s_4 \in S_k, \\ s_1 s_2 \in \delta(S) \text{ and } s_3 s_4 \in \delta(S) \\ \text{for some } \rho \in \mathbb{R}. \end{array} \tag{79}$$

Next, we will establish the coefficients between terminals.

Consider a salesman $l \in K$ and let w_i and w_{i+1} be two terminals of T_l. Let s be a Steiner node of S_l and consider the solution (U^8, I^8) given by $U_l^8 = (U_l^0 \backslash \{w_i w_{i+1}\}) \cup$

$\{w_is, sw_{i+1}\}$, $U^8_p = U^0_p$, for each $p \in \{1, \ldots, |K|\} \setminus \{l\}$ and $I^8 = \bigcup_{p\in K} U^8_p$. It is clear that when $l \neq k$, the incidence vector of solution (U^8, I^8) is in $F^k_{j,S}$. Note here that when $l = k$, one can also easily check that (x^{U^8}, y^{I^8}) is in $F^k_{j,S}$ for all choice of the Steiner node s (either $s \in S$ or $s \in S_k \setminus S$) and for every choice of terminals w_i and w_{i+1} (in particular w_iw_{i+1} can be equal to w_jw_{j+1} or $w_{j-1}w_j$). As (x^{U^8}, y^{I^8}) is in $F^k_{j,S}$ and hence in F, it follows that $rx^{U^0} = rx^{U^8} = rx^{U^0} - r^l_{w_iw_{i+1}} + r^l_{w_is} + r^l_{sw_{i+1}}$, implying that $r^l_{w_iw_{i+1}} = r^l_{w_is} + r^l_{sw_{i+1}}$. salesman l and nodes s, w_i and w_{i+1} are all arbitrary. Therefore

$$r^l_{w_iw_{i+1}} = r^l_{w_is} + r^l_{sw_{i+1}} \qquad \text{for all } w_i, w_{i+1} \in T_l,\ s \in S_l,\ l \in K. \tag{80}$$

In what follows, we will look at the coefficients of edges between terminals and Steiner nodes.

Consider a salesman $l \in K \setminus \{k\}$ and let (U^9, I^9) be the pair obtained from (U^8, I^8) as follows, $U^9_l = (U^8_l \setminus \{sw_{i+1}\}) \cup \{ss', s'w_{i+1}\}$, $U^9_p = U^8_p$ for all $p \in \{1, \ldots, |K|\} \setminus \{l\}$ and $I^9 = \bigcup_{j\in K} U^9_j$, where s and s' are Steiner nodes of S_l. Since (x^{U^8}, y^{I^8}) and (x^{U^9}, y^{I^9}) are both in $F^k_{j,S}$ and thus in F, this implies that $rx^{U^8} = rx^{U^9} = rx^{U^8} - r^l_{sw_{i+1}} + r^l_{ss'} + r^l_{s'w_{i+1}}$. By (77), it follows that $r^l_{sw_{i+1}} = r^l_{s'w_{i+1}}$. As salesman l, and nodes w_i, s and s' are arbitrary, we have

$$r^l_{sw_i} = r^l_{s'w_i} = \lambda^l_1(w_i) \qquad \begin{array}{l}\text{for all } w_i \in T_l,\ s, s' \in S_l,\ l \in K \setminus \{k\},\\ \text{for some } \lambda^l_1(w_i) \in \mathbb{R}.\end{array} \tag{81}$$

Consider two Steiner nodes s and s' of S_k such that $ss' \notin \delta(S)$. Let (U^{10}, I^{10}) be the solution such that $U^{10}_k = (U^0_k \setminus \{w_iw_{i+1}\}) \cup \{w_is, sw_{i+1}\}$, $U^{10}_p = U^0_p$, for all $p \in \{1, \ldots, |K|\} \setminus \{k\}$ and $I^{10} = \bigcup_{p\in K} U^9_p$. We also define (U^{11}, I^{11}) by $U^{11}_k = (U^{10}_k \setminus \{w_is\}) \cup \{w_is', ss'\}$, $U^{11}_p = U^{10}_p$, for all $p \in \{1, \ldots, |K|\} \setminus \{k\}$ and $I^{11} = \bigcup_{p\in K} U^{11}_p$. Note that $(x^{U^{10}}, y^{I^{10}})$ and $(x^{U^{11}}, y^{I^{11}})$ are both in $F^k_{j,S}$ and thus in F. As a consequence, we have $rx^{U^{10}} = rx^{U^{10}} - r^k_{w_is} + r^k_{w_is'} + r^k_{ss'}$. As by (78), $r^k_{ss'} = 0$, we have $r^k_{w_is} = r^k_{w_is'}$. And since w_i, s and s' are arbitrary, we can write

$$r^k_{w_is} = r^k_{w_is'} = \lambda^k_1(w_i) \qquad \begin{array}{l}\text{for all } w_i \in T_k,\ s, s' \in S_k,\ ss' \notin \delta(S),\\ \text{for some } \lambda^l_1(w_i) \in \mathbb{R}.\end{array} \tag{82}$$

If $ss' \in \delta(S)$, we can similarly show that $r^k_{w_is} = r^k_{w_is'} + r^k_{ss'}$. Since w_i, s and s' are arbitrary, we have

$$r^k_{st_i} = r^k_{s't_i} + r^k_{ss'} = \lambda^l_1(w_i) + \rho \qquad \begin{array}{l}\text{for all } w_i \in T_k,\ s, s' \in S_k,\ ss' \notin \delta(S),\\ \text{for some } \lambda^l_1(w_i) \in \mathbb{R},\ \rho \in \mathbb{R}.\end{array} \tag{83}$$

Now, let $\rho \in \mathbb{R}$ be as given by (79) and $\lambda^l = (\lambda_1^l, \lambda_2^l)$, $l \in K$ such that $\lambda_1^l = (\lambda_1^l(w_i), w_i \in T_k)$ where $\lambda_1^l(w_i)$ is as given by (81), (82) and (83). $\lambda_2^l = (\lambda_2^l(uv),\ u, v \in T_k, uv \notin U_l^0)$ such that $\lambda_2^l(uv) = r_{uv}^l - \lambda_1^l(u) - \lambda_1^l(v), l \in K$.

Overall, the coefficients r_{uv}^l for all $uv \in E$ and $l \in K$ can then be expressed in terms of ρ, λ_1^l and λ_2^l as follows

$$r_{uv}^l = \begin{cases} \lambda_1^l(u) + \lambda_1^l(v) & \text{if } l \neq k,\ uv = w_i w_{i+1},\ w_i, w_{i+1} \in T_l, \\ \lambda_1^l(u) & \text{if } l \neq k,\ u \in T_l,\ v \in S_l, \\ \lambda_1^k(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \notin \delta(S) \\ \rho + \lambda_1^k(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \in \delta(S) \\ \lambda_2^l(uv) + \lambda_1^l(u) + \lambda_1^l(v) & \text{if } uv = w_i w_j,\ w_i, w_j \in T_l,\ j > i, \\ 0 & \text{if } l \neq k \text{ and } uv = s_i s_j,\ s_i, s_j \in T_l,\ j \neq i, \\ 0 & \text{if } l = k \text{ and } uv = s_i s_j,\ s_i, s_j \in S_k,\ uv \notin \delta(S), \\ \rho & \text{if } l = k \text{ and } uv = s_i s_j,\ s_i, s_j \in S_k,\ uv \in \delta(S). \end{cases}$$

It is clear that $r^l = \rho a^l + \lambda_1^l M_1^l + \lambda_2^l M_2^l,$ for all $l \in K$. We then deduce that $r = \rho a + \lambda M$ and the result follows.

B.3 Proof of Claim 4.3

If edge f is considered in some solution, there are two possible configurations.

- The first corresponds to the case where f is not used to route some section of salesman k. Consider solution (U^1, I^1) defined as follows: $U_k^1 = U^0 \cap f$ and $U_j^1 = U_j^0$, $j = 1, \ldots, |K|$, $j \neq k$, and $I^1 = \bigcup_{j \in K} U_j^1$. In this case, it is not hard to see that $x^{U^1,k}(\delta(V_i)) \geq 3$ and $x^{U^1,k}(\delta(V_j)) \geq 3$ and hence $x^{U^1,k}(\delta(V_i)) + x^{U^1,k}(\delta(V_j)) \geq 6$.
- The second configuration is when edge f is used to route some section of salesman k. Suppose, without loss of generality, that this section is (t_i, t_{i+1}), where $t_i \in V_i$ and $t_{i+1} \notin (V_i \cap V_j)$. Consider solution (U^2, I^2) defined as follows: $U_k^2 = U^0 \backslash t_i t i + 1 \cap \{t_i s, ss', s' t_{i+1}\}$ and $U_j^2 = U_j^0$, $j = 1, \ldots, K$, $j \neq k$, and $I^2 = \bigcup_{j \in K} U_j^1$. Clearly, the following hold for solution (U^2, I^2): $x^{U^2,k}(\delta(V_i)) \geq 2$ and $x^{U^2,k}(\delta(V_j)) \geq 4$. And this implies that $x^{U^2,k}(\delta(V_i)) + x^{U^1,k}(\delta(V_j)) \geq 6$.

B.4 Proof of Theorem 9

Denote inequality (27) by $ax + by \leq \alpha$. Let $rx + qy \leq \beta$ be a valid inequality defining a facet F of MSTSPOC(G, K, T) such that $F_{\Pi,F}^k \subseteq F$, where $F_{\Pi,F}^k$ is the face induced by inequality (27). In what follows, we prove that there exist $\rho \in \mathbb{R}$ and $\lambda = (\lambda^l, l \in K)$, $\lambda^l \in \mathbb{R}^{|T_l| + p_l}$ for $l \in K$, such that $q = \rho b$ and $r = \rho a + \lambda M$. Notice here that $a = (a^1, a^2, \ldots, a^{|K|})$ such that $a^i \in \mathbb{R}^m$, $i = 1, \ldots, |K|$ with $a^i = 0$ for $i \in \{1, \ldots, |K|\} \backslash \{k\}$, $a_e^k \neq 0$ for every $e \in \delta(\Pi) \backslash F$ and $a_{e'}^k = 0$ for every $e' \in E \backslash (\delta(\Pi) \backslash F)$. Note also that $b = 0$.

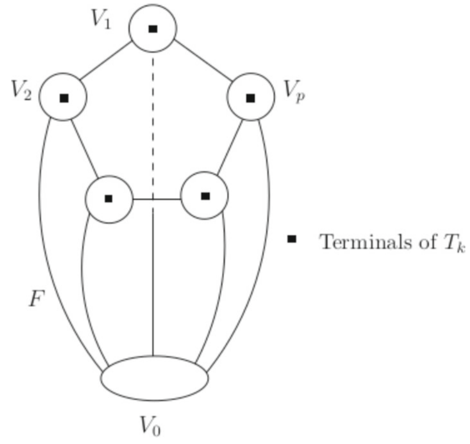

Fig. 12 An odd wheel configuration

Throughout the proof, we will suppose for convenience that the edges of F are linking Steiner nodes of V_0 to terminals of $V_i, i = 1, \ldots, p$. Moreover, we will restrict ourselves to the case where p is odd, the case where p is even can be done along the same line. Remark that, under this hypothesis, we have an odd-wheel configuration as shown in Fig. 12.

In the sequel, we will suppose that the terminals of $V_1, V_2, \ldots, V_p$ are consecutive. We will refer to the terminals of these sets by $t_1, t_2, \ldots, t_p$. In addition, we will denote the edges of F by $f_1, f_2, \ldots, f_p$, where $f_i = t_i s_i, i = 1, \ldots, p$.

First, we prove that $q = 0$.

Consider the solution (U^0, I^0) and let $e_1, e_2, \ldots, e_p$ be the edges between consecutive terminals ($e_i = t_i t_{i+1}$, with $t_{p+1} = t_1$). Note that $U^0 = \{e_1, \ldots, e_p\}$.

Let (U^1, I^1) be the solution given by

$U_k^1 = \{e_1, e_2, f_3, s_3 s_4, f_4, e_4, f_5, \ldots, e_{p-1}, f_p, s_{p-1} s_p, f_1\}$ and $U_j^1 = U_j^0, j = 1, \ldots, K, j \neq k$, and $I^1 = \bigcup_{j \in K} U_j^1$ (note here that one can have $s_3 = s_4$ and/or $s_{p-1} = s_p$). Consider an edge $g \in E \setminus I^1$ and let (U^2, I^2) be the solution given by $U^2 = U^1$ and $I^2 = I^1 \cup \{g\}$. It is not hard to see that the incidence vectors of solutions (U^1, I^1) and (U^2, I^2) are in $F^k_{\Pi,F}$. This means that (x^{U^1}, y^{I^1}) and (x^{U^2}, y^{I^2}) satisfy equation $rx + qy = \beta$. Consequently, $rx^{U^1} + qy^{I^1} = rx^{U^2} + qy^{I^2} = qx^{U^1} + qy^{I^1} + q_e$, implying that $q_e = 0$. Since e is arbitrarily chosen in $E \setminus I^1$, we then have that

$$q_e = 0 \qquad \text{for all } e \in E \setminus I^1. \tag{84}$$

Now consider an edge g_1 of I^1 and a salesman $l \in K$.

Here we distinguish two cases.

First, suppose that $l \neq k$. Hence $g_1 \in U_l^0$. Let $g_1 = t_i t_{i+1}$ and (U^3, I^3) the solution given by $U_l^3 = (U_l^1 \setminus \{g_1\}) \cup \{t_i s, s t_{i+1}\}$, where $s \in S_l$ is a Steiner node of salesman l. In addition, let $U_j^3 = U_j^1, j = 1, \ldots, K, j \neq l$, and $I^3 = \bigcup_{j \in K} U_j^3$. Let (U^4, I^4) given by $U^4 = U^3$ and $I^4 = I^3 \cup \{g_1\}$. The incidence vectors of both solutions

(U^3, I^3) and (U^4, I^4) are in $F^k_{\Pi,F}$, and hence $rx + qy = \beta$. As a consequence, we have $rx^{U^3} + qy^{I^3} = rx^{U^4} + qy^{I^4} = rx^{U^3} + qy^{I^3} + q_{g_1}$, which implies that $q_{g_1} = 0$.

Suppose now that $l = k$. Here also, we distinguish two cases. The first one is when $g_1 \in I^1 \cap I^0$. This is similar to the previous one. The second one is when $g_1 \in I^1 \setminus I^0$, that $g_1 \in I^1 \cap F$. Suppose, without loss of generality, that $g_1 = f_1$. Let (U^5, I^5) be the solution such that $U^5_l = (U^1_l \setminus \{f_1, e_2\}) \cup \{s_p t_1, f_2, s_1 s_2\}$. Also, let $U^5_j = U^1_j$, $j = 1, \ldots, K$, $j \neq l$, and $I^5 = \bigcup_{j \in K} U^5_j$. Let (U^6, I^6) given by $U^6 = U^5$ and $I^6 = I^5 \cup \{f_1\}$. It is clear that (x^{U^5}, y^{I^5}) and (x^{U^6}, y^{I^6}) are in $F^k_{\Pi,F}$ and hence in F. Thus, we have $rx^{U^5} + qy^{I^5} = rx^{U^6} + qy^{I^6} = rx^{U^5} + qy^{I^5} + q_{f_1}$, which implies that $q_{f_1} = 0$.

As g_1 is arbitrary in I^1, this implies that

$$q_e = 0 \qquad \text{for all } e \in I^1. \tag{85}$$

By (84) and (85) we then have

$$q_e = 0 \qquad \text{for all } e \in E. \tag{86}$$

In what follows, we will examine components of vector r.

First, we will establish the coefficients of edges between Steiner nodes.

Consider a salesman $l \in K \setminus \{k\}$ and let s and s' be two Steiner nodes of S_l. Consider the solution (U^1, I^1) defined above and let (U^7, I^7) be the solution defined by $U^7_l = U^1_l \cup \{ss'\}$, $U^7_p = U^1_p$ for all $p \in \{1, \ldots, |K|\} \setminus \{l\}$ and $I^7 = \bigcup_{p \in K} U^7_p$. Clearly (x^{U^1}, y^{I^1}) and (x^{U^7}, y^{I^7}) are both in $F^k_{\Pi,F}$ and thus in F. This implies that (x^{U^1}, y^{I^1}) and (x^{U^7}, y^{I^7}) satisfy equation $rx + qy = \beta$. Since $q = 0$, we have $rx^{U^1} = rx^{U^7} = rx^{U^1} + r^l_{ss'}$, which implies $r^l_{ss'} = 0$. As salesman l and Steiner nodes s and s' are arbitrary, we have

$$r^l_{ss'} = 0 \qquad \text{for all } \ s, s' \in S_l, \ l \in K \setminus \{k\}. \tag{87}$$

Now consider salesman k and suppose that s and s' are Steiner nodes of S_k. Here, we shall distinguish different cases. First, suppose that $ss' \in V_j$ for a given $j \in \{1, \ldots, p\}$. Along the same line, we can prove that $r^k_{ss'} = 0$.

Similarly, we can also prove that $r^k_{ss'} = 0$ for every s and s' of $S_k \cap V_0$ such that $|\delta(s) \cap F| + |\delta(s') \cap F| \leq 1$.

Now suppose that s and s' are in $S_k \cap V_0$ but $\delta(s) \cap F \neq \emptyset$ and $\delta(s') \cap F \neq \emptyset$. Suppose, without loss of generality, that $s = s_p$ and $s' = s_1$. Consider again solution (U^1, I^1) and define solution (U^8, I^8) as follows, $U^8_k = U^1_k \cup \{s_p s_1, s_p s_2, s_2 s_1\}$, $U^8_p = U^1_p$ for all $p \in \{1, \ldots, |K|\} \setminus \{k\}$ and $I^8 = \bigcup_{p \in K} U^8_p$. It is clear that (x^{U^1}, y^{I^1}) and (x^{U^8}, y^{I^8}) are both in $F^k_{\Pi,F}$ and thus in F. This implies that $rx^{U^1} = rx^{U^8} = rx^{U^1} - r^k_{s_p s_1} + r^k_{s_p s_2} + r^k_{s_2 s_1}$, and hence $r^k_{s_p s_1} = r^k_{s_p s_2} + r^k_{s_2 s_1}$. By symmetry, we have

$$\begin{array}{rcl} r^k_{s_1s_2} & = & r^k_{s_2s_3} + r^k_{s_3s_1}, \\ r^k_{s_2s_3} & = & r^k_{s_3s_4} + r^k_{s_4s_2}, \\ \vdots & & \\ r^k_{s_ps_1} & = & r^k_{s_ps_2} + r^k_{s_2s_1}, \end{array}$$

yielding to $r^k_{ss'} = 0$ for all Steiner nodes s and s' in $S_k \cap V_0$ whose incident edges intersect F.

Overall, we have

$$r^k_{ss'} = 0 \qquad \text{for all } \; s, s' \in S_k, \; ss' \notin \delta(\Pi). \tag{88}$$

Now, suppose that s and s' are two Steiner nodes such that $s \in V_j$ and $s' \in V_{j+1}$ for some $j \in \{1, \ldots, p\}$. Suppose also that there is a Steiner node $s'' \in V_{j+1}$. Notice that ss' and ss'' are both in $\delta(\Pi)\backslash F$.

Suppose, without loss of generality, that $j = 1$ (the result can be found by symmetry for all the sets V_i). Consider solution (U^1, I^1) given above and define solution (U^9, I^9) as follows, $U^9_k = U^1_k\backslash\{e_1\} \cup \{t_1s, ss', s't_2\}$, $U^9_p = U^0_p$ for all $p \in \{1, \ldots, |K|\}\backslash\{k\}$ and $I^9 = \bigcup_{p\in K} U^9_p$. Let (U^{10}, I^{10}) be the solution defined by $U^{10}_k = U^9_k\backslash\{ss'\} \cup \{ss'', s''s'\}$, $U^{10}_p = U^9_p$ for all $p \in \{1, \ldots, |K|\}\backslash\{k\}$ and $I^{10} = \bigcup_{p\in K} U^{10}_p$. Clearly, (x^{U^9}, y^{I^9}) and $(x^{U^{10}}, y^{I^{10}})$ are both in $F^k_{\Pi,F}$ and then satisfy equation $rx + qy = \beta$. Since $q = 0$, we have $rx^{U^9} = rx^{U^{10}} = r^k x^{U^9} - r^k_{ss'} + r^k_{ss''} + r^k_{s''s'}$. By (88), we have $r^k_{s''s'} = 0$, which implies that $r^k_{ss'} = r^k_{ss''}$. As Steiner nodes s, s' and s'' are arbitrary of S_k, we have

$$r^k_{ss'} = r^k_{ss''} = \rho \qquad \begin{array}{l} \text{for all } \; s, s', s'' \in S_k, \; ss', ss'' \in \delta(\Pi) \setminus F, \\ \text{for some } \rho \in \mathbb{R}. \end{array} \tag{89}$$

Now, we will determine the coefficients of edges between terminals.

Consider a salesman $l \in K \setminus \{k\}$ and let t_i and t_{i+1} be two terminals of T_l. Let s be a Steiner node of S_l and consider the solution (U^{11}, I^{11}) given by $U^{11}_l = (U^1_l\backslash\{t_it_{i+1}\}) \cup \{t_is, st_{i+1}\}$, $U^{11}_p = U^1_p$, for each $p \in \{1, \ldots, |K|\}\backslash\{l\}$ and $I^{11} = \bigcup_{p\in K} U^{11}_p$. Clearly, the incidence vector of solution (U^{11}, I^{11}) is in $F^k_{\Pi,F}$ and hence in F. It follows that $rx^{U^1} = rx^{U^{11}} = rx^{U^1} - r^l_{t_it_{i+1}} + r^l_{t_is} + r^l_{st_{i+1}}$, implying that $r^l_{t_it_{i+1}} = r^l_{t_is} + r^l_{st_{i+1}}$. salesman l and nodes s, t_i and t_{i+1} are all arbitrary. Therefore

$$r^l_{t_it_{i+1}} = r^l_{t_is} + r^l_{st_{i+1}} \qquad \text{for all } t_i, t_{i+1} \in T_l, \; s \in S_l, \; l \in K \setminus \{k\}. \tag{90}$$

Now consider salesman k and suppose that t_i and t_{i+1} are terminals of T_k, such that t_i and t_{i+1} are in V_i and V_{i+1}, respectively. Suppose, without loss of generality, that $t_i = t_1$ and $t_{i+1} = t_2$. Consider solution (U^1, I^1) and let (U^{12}, I^{12}) be the solution defined as follows, $U^{12}_k = (U^1_k\backslash\{t_1t_2\}) \cup \{t_1s_2, s_2t_2\}$, $U^{12}_p = U^1_p$, for each $p \in \{1, \ldots, |K|\}\backslash\{k\}$ and $I^{12} = \bigcup_{p\in K} U^{12}_p$. (x^{U^1}, y^{I^1}) and $(x^{U^{12}}, y^{I^{12}})$ are both in $F^k_{\Pi,F}$ and hence in F. Therefore, they satisfy equation $rx + qy = \beta$. This implies that

$rx^{U^1} = rx^{U^{12}} = rx^{U^1} - r^k_{t_it_{i+1}} + r^k_{t_is_2} + r^k_{s_2t_{i+1}}$. Hence $r^k_{t_it_{i+1}} = +r^k_{t_is_2} + r^k_{s_2t_{i+1}}$. As a consequence, we have

$$r^k_{t_it_{i+1}} = r^k_{t_is} + r^k_{st_{i+1}} \quad \begin{array}{l}\text{for all } t_i, t_{i+1} \in T_k,\ s \in S_k,\\ t_it_{i+1}, t_is \in \delta(\Pi),\ \text{and } st_{i+1} \in F.\end{array} \tag{91}$$

In what follows, we will examine at the coefficients of edges between terminals and Steiner nodes.

Consider a salesman $l \in K \setminus \{k\}$ and two Steiner nodes s and s' of S_l. Let (U^{13}, I^{13}) be the solution obtained from (U^7, I^7) as follows, $U^{13}_l = (U^7_l \backslash \{st_{i+1}\}) \cup \{ss', s't_{i+1}\}$, $U^{13}_p = U^7_p$ for each $p \in \{1, \ldots, |K|\} \backslash \{l\}$ and $I^{13} = \bigcup_{j \in K} U^{13}_j$. Since (x^{U^7}, y^{I^7}) and $(x^{U^{13}}, y^{I^{13}})$ are both in $F^k_{\Pi,F}$ and thus in F, this implies $rx^{U^7} = rx^{U^{13}} = rx^{U^7} - r^l_{st_{i+1}} + r^l_{ss'} + r^l_{s't_{i+1}}$. By (87), it follows that $r^l_{st_{i+1}} = r^l_{s't_{i+1}}$. As salesman l, and nodes t_i, s and s' are arbitrary, we have

$$r^l_{st_i} = r^l_{s't_i} = \lambda^l_1(t_i) \quad \begin{array}{l}\text{for all } t_i \in T_l,\ s, s' \in S_l,\ l \in K \setminus \{k\},\\ \text{for some } \lambda^l_1(t_i) \in \mathbb{R}.\end{array} \tag{92}$$

Now, consider salesman k. Consider a terminal t_i of T_k and Steiner nodes s and s' of S_k such that t_i, s and s' belong to the same set, say V_i.

Along the same line, we can prove that

$$r^k_{st_i} = r^k_{s't_i} = \lambda^k_1(t_i) \quad \begin{array}{l}\text{for all } t_i \in T_k,\ s, s' \in S_k,\\ st_i, s't_i \notin \delta(\Pi),\\ \text{for some } \lambda^k_1(t_i) \in \mathbb{R}.\end{array} \tag{93}$$

Now, suppose that t_i, s and s' are such that $t_is \notin \delta(\Pi)$ and $t_is' \in \delta(\Pi) \backslash F$. Without loss of generality, we will suppose that $t_i = t_1$ and $s' = s_2$. Consider solution (U^{14}, I^{14}) defined as follows. $U^{14}_k = (U^1_k \backslash \{t_1t_2\}) \cup \{t_1s_2, s_2t_2\}$, $U^{14}_p = U^1_p$, for each $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^{14} = \bigcup_{p \in K} U^{14}_p$. Define also solution (U^{15}, I^{15}) given by $U^{15}_k = (U^{14}_k \backslash \{t_1s_2\}) \cup \{t_1s, ss_2\}$, $U^{15}_p = U^{14}_p$, for each $p \in \{1, \ldots, |K|\} \backslash \{k\}$ and $I^{15} = \bigcup_{p \in K} U^{15}_p$. Since $(x^{U^{14}}, y^{I^{14}})$ and $(x^{U^{15}}, y^{I^{15}})$ are both in $F^k_{\Pi,F}$ and thus in F, this implies $rx^{U^{14}} = rx^{U^{15}} = rx^{U^{14}} - r^k_{t_1s_2} + r^k_{t_1s} + r^k_{ss_2}$. As t_i, s and s' are all arbitrary, we have

$$r^k_{st_i} = r^k_{s't_i} + r^k_{ss'} = \lambda^l_1(t_i) + \rho \quad \begin{array}{l}\text{for all } t_i \in T_k,\ s, s' \in S_k,\\ t_is' \notin \delta(\Pi),\ \text{and } t_is, ss' \in \delta(\Pi) \setminus F,\\ \text{for some } \lambda^k_1(t_i) \in \mathbb{R},\ \rho \in \mathbb{R}.\end{array} \tag{94}$$

Now, let $\rho \in \mathbb{R}$ be as given by (89) and $\lambda^l = (\lambda^l_1, \lambda^l_2)$, $l \in K$ such that $\lambda^l_1 = (\lambda^l_1(t_i), t_i \in T_k)$ where $\lambda^l_1(t_i)$ is as given by (92), (93) and (94). $\lambda^l_2 = (\lambda^l_2(uv),\ u, v \in T_k, uv \notin U^0_l)$ such that $\lambda^l_2(uv) = r^l_{uv} - \lambda^l_1(u) - \lambda^l_1(v)$, $l \in K$.

Overall, the coefficients r^l_{uv} for all $uv \in E$ and $l \in K$ can then be expressed in terms of ρ, λ^l_1 and λ^l_2 as follows

$$r^l_{uv} = \begin{cases} \lambda^l_1(u) + \lambda^l_1(v) & \text{if } l \in K,\ uv = t_i t_{i+1},\ t_i, t_{i+1} \in T_l, \\ \lambda^l_1(u) & \text{if } l \neq k,\ u \in T_l,\ v \in S_l, \\ \lambda^k_1(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \notin \delta(\Pi), \\ \lambda^k_1(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \in F, \\ \rho + \lambda^k_1(u) & \text{if } l = k,\ u \in T_k,\ v \in S_k, uv \in \delta(\Pi) \setminus F, \\ \lambda^l_2(uv) + \lambda^l_1(u) + \lambda^l_1(v) & \text{if } uv = t_i t_j,\ t_i, t_j \in T_l,\ j > i, \\ 0 & \text{if } l \neq k \text{ and } uv = s_i s_j,\ s_i, s_j \in T_l,\ j \neq i, \\ 0 & \text{if } l = k \text{ and } uv = s_i s_j,\ s_i, s_j \in S_k,\ uv \notin \delta(\Pi), \\ \rho & \text{if } l = k \text{ and } uv = s_i s_j,\ s_i, s_j \in S_k,\ uv \in \delta(\Pi) \setminus F. \end{cases}$$

It is clear that $r^l = \rho a^l + \lambda^l_1 M^l_1 + \lambda^l_2 M^l_2$, for all $l \in K$. We then have that $r = \rho a + \lambda M$ and the result follows.

C Appendix: Further valid Inequalities

Proposition 8 *Consider a salesman $k \in K$ and let $\Pi = (V_1, ..., V_p)$ be a partition of V such that $|V_i \cap T_k| \geq 1, i = 1, ..., p$ ($p \geq 2$). Suppose that $V_1, ..., V_r$, $r \leq p$ contain respectively $q_i \geq 2$, $i = 1, ..., r$ non-successive terminals (or sequences of terminals). Let $S \subseteq S_k$ be a subset of Steiner nodes of salesman k. Inequalities (29) are redundant with respect to inequalities (24), (3) and (5).*

Proof Denote by $V_{i,j}$, $i = 1, ..., p$ and $j = 1, ..., q_i$, the j^{th} component of the set V_i consisting of only one terminal or a sequence of successive terminals. Clearly, the following inequalities are valid for MSTSPOC(G, K, T)

$$\begin{array}{rll} x^k(\delta(V_{i,j})) & \geq 2 & \text{for all } i = 1, ..., p,\ j = 1, ..., q_i \\ -x^k(\delta(s)) & \geq -2 & \text{for all } s \in S, \\ x^k(e) & \geq 0 & \text{for all } e \in E(S_k) \setminus E(S). \end{array}$$

Remark that there are exactly $p + \sum_{i=1}^r q_i - r$ Steiner cut inequalities and $|S|$ disjunction inequalities.

By summing these inequalities, together with (10), we obtain

$$2x^k(\delta_{G \setminus S}(V_1, V_2, ..., V_p)) \geq 2((p + \sum_{i=1}^r q_i - r) - |S|),$$

and by dividing by 2, we obtain

$$x^k(\delta_{G \setminus S}(V_1, V_2, ..., V_p)) \geq (p + \sum_{i=1}^r q_i - r) - |S|,$$

which ends the proof. □

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